The Künneth theorem in Homotopy Type Theory

Supervisors: Pierre Boutry, Viktoria Heu, Loïc Pujet

Background

Algebraic topology

Algebraic topology is a branch of mathematics in which one studies spaces by associating algebraic objects to them, such that two spaces which are homotopically equivalent (intuitively, equivalent up to continuous deformations) receive isomorphic algebraic objects. This programme has been very successful in helping us understand spaces up to homotopy, and the methods of algebraic topology are commonly used in almost all of the branches of mathematics.

The two most well-known algebraic invariants are *homotopy groups* and *homology groups*. The nth homotopy group of a space X is noted $\pi_n(X)$, and it is defined as the group of homotopy classes of mappings from the n-dimensional sphere $\mathbb{S}^n$ to X. This definition is conceptually simple, but unfortunately it turns out that homotopy groups are very difficult to compute for $n > 1$. In fact, we do not even know all the homotopy groups of the 2-dimensional sphere! This is why, in practice, most computations are done with the other family of invariants, the homology groups $\widetilde{H}_n(X, R)$ (where R is any ring). These are somewhat similar to homotopy groups in that they intuitively measure the number of n-dimensional holes, but their definition in terms of linear algebra makes them much more approachable.

Homotopy type theory

Homotopy type theory [7] (HoTT) is a young subfield of type theory that incorporates ideas from homotopy theory. It starts with the postulate that a proof of equality between two types is the same thing as an isomorphism, which naturally leads to a theory with a homotopical flavour. Types behave as spaces, terms as points, equality proofs as paths, *etc.*

In fact, these homotopical aspects are very concrete: it becomes possible to define the homotopy groups of a type, and to reformulate a large portion of classical algebraic topology in terms of types, terms and equality proofs. For instance, Brunerie proved that the fourth homotopy group of the three-dimensional sphere is equal to $\mathbb{Z}/2\mathbb{Z}$ [1]. Of course, homotopy groups remain just as difficult to compute as in classical algebraic topology, if not more. Thus, it is natural to try and reproduce the theory of homology groups, which are usually much easier to work with.

There are two definitions of homology in HoTT. The first one is due to Christensen and Scoccola and uses *spectra* [4], while the other, due to Ljungström and Pujet, relies on the cellular structure of CW complexes [2, 5].

The Künneth theorem

One of the most basic ways of constructing new spaces is to form *cartesian products*. Now, the homotopy groups of a cartesian product are very simple to compute. Indeed, as a straightforward consequence of their definition, we have $\pi_n(X \times Y) \simeq \pi_n(X) \times \pi_n(Y)$ [6]. However, the same cannot be said for homology! In order to express the homology groups of a cartesian product, we first need to define the tensor product of two $\mathbb{N}$-indexed families of R-modules X_* and Y_*:

$$(X_* \otimes Y_*)_n := \bigoplus_{i+j=n} X_i \otimes_R X_j.$$

Then, in the special case where the ring of coefficients R is a field, we have

$$\widetilde{H}_*(X \times Y, R) \simeq \widetilde{H}_*(X, R) \otimes \widetilde{H}_*(Y, R).$$

In case R is not a field but only a principal ideal domain (PID), we have to add corrective factors called the Tor groups [9], whose definition involves more sophisticated commutative algebra. More precisely, we get the following short exact sequence for any n:

$$0 \longrightarrow \left(\widetilde{H}_*(X, R) \otimes \widetilde{H}_*(Y, R)\right)_n \longrightarrow \widetilde{H}_n(X \times Y, R) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(\widetilde{H}_i(X, R), \widetilde{H}_j(Y, R)) \longrightarrow 0.$$

Goals of the internship

The goal of the internship is to prove the Künneth theorem in homotopy type theory, and to formalise the result in Cubical Agda [8]. The easiest option is to work with CW complexes, in which case we can compute the cellular chain complex of a cartesian product to be a tensor product of cellular chain complexes:

$$C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y).$$

From there, we can deduce the Künneth formula for $\widetilde{H}_*(X \times Y, R)$, at least when R is a field.

The general form of the Künneth theorem, where R is only assumed to be PID, involves the Tor groups. In order to transpose this theorem in HoTT without having to make extra assumptions, we need to find an appropriate (in particular, constructive) definition for the Tor groups, which does not yet exist in the literature – as of today, there only exists a constructive version of the closely related Ext groups [3]. Another possible generalisation is to work with Christensen and Scoccola's definition for homology based on spectra.

Required skills

The intern is expected to have some experience with proof assistants and with type theory (ideally, the intern has completed a course in homotopy type theory). There are no requirements concerning algebraic topology, although some familiarity with the topic would surely be helpful.

Organization

The internship will take place in the University of Strasbourg, where Pierre Boutry, Viktoria Heu and Loïc Pujet are based.

References

[1] Guillaume Brunerie. On the homotopy groups of spheres in homotopy type theory, 2016.

[2] Ulrik Buchholtz and Kuen-Bang Hou. Cellular Cohomology in Homotopy Type Theory. *Logical Methods in Computer Science*, Volume 16, Issue 2, June 2020.

[3] J. Daniel Christensen and Jarl G. Taxerås Flaten. Ext groups in Homotopy Type Theory, 2025.

[4] J. Daniel Christensen and Luis Scoccola. The Hurewicz theorem in Homotopy Type Theory. *Algebraic & Geometric Topology*, 23(5):2107–2140, 2023.

[5] Axel Ljungström and Loïc Pujet. Cellular Methods in Homotopy Type Theory, 2025.

[6] Jon Peter May. *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics. University of Chicago Press, 1999.

[7] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[8] Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types. *Proc. ACM Program. Lang.*, 3(ICFP), July 2019.

[9] Charles Alexander Weibel. *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.