Cellular Methods in Homotopy Type Theory

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***Abstract*—In classical mathematics, a CW complex is a topological space which can be built up inductively by gluing together cells of increasing dimension. Due to their good categorical properties, CW complexes have become the main object of interest in algebraic topology. Although their quasi-combinatorial nature suggests that a constructive treatment is possible, there seems to be little literature on the subject – perhaps because of the important role played by the axiom of choice in the classical theory of CW complexes.**

In this paper, we present a *synthetic* and *constructive* account of the theory of CW complexes in homotopy type theory. Most notably, we prove a finitary version of the cellular approximation theorem, which allows us to construct a theory of cellular homology without needing the axiom of choice or relying on a pre-existing notion of homology. We prove that our cellular homology is functorial and that it satisfies a finitary version of the Eilenberg-Steenrod axioms. Last but not least, we give a constructive proof of the Hurewicz theorem, which relates the first non-vanishing homotopy group of a CW complex with the corresponding homology group. All theorems presented in this paper have been mechanised in Cubical Agda.

I. Introduction

Homotopy type theory (HoTT) is an extension of intensional type theory that is built around an analogy between type theory and homotopy theory [1]. It provides a synthetic framework for reasoning about spaces and their homotopy invariants, and has been successfully used to formalise a number of results from algebraic topology [2]. In this paper, we present a development of the theory of CW complexes in HoTT, including cornerstone results such as the cellular approximation theorem, cellular homology, and the Hurewicz theorem.

An important aspect of homotopy type theory is that it is fully constructive. In particular, neither the law of excluded middle nor the axiom of choice are available, which means that we have to reformulate many of our theorems in ways which make them constructively provable. In return for these extra efforts, the theorems we prove are more general than the corresponding theorems from classical algebraic topology: since HoTT has models in all ∞-toposes [3], [4], the developments presented in this paper effectively show that cellular methods are available in this very general setting.

A. Outline and contributions

In Section II, we outline the basic definitions from HoTT which we will need for our development, with special emphasis on homotopy pushouts and truncations.

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In Section III, we develop our constructive theory of CW complexes. Our main contributions are a construction of the pushout of two cellular maps (Definition 9 and Proposition 10), and a constructive treatment of the cellular approximation theorem for maps and homotopies (Theorems 14 and 21). We define various categories which, to a varying degree, capture the idea of a cellular space in HoTT, and we study the relations between these categories.

In Section IV, we apply our results to the construction of homology theories for our categories of cellular spaces. Our basic definition of homology groups comes from Buchholtz and Favonia [5], but we give novel proofs for functoriality and homotopy invariance: instead of relying on a pre-existing notion of homology, we use our freshly proved cellular approximation theorem. Finally, we verify that our homology functors satisfy the Eilenberg-Steenrod axioms. We remark that most proofs and constructions in this section can be interpreted in the setting of cellular *co*homology. We simply chose to focus on homology because it constituted an open problem.

In Section V, we prove the Hurewicz theorem for our homology theory. To this end, we prove a special case of the so called *CW approximation theorem* which shows that, for CW complexes, the usual definition of an n-connected types in HoTT coincides with the classical definition of an n-connected CW complex (Corollary 47). We emphasise that the approximation theorems that underlie this work are results whose classical proofs tend to be inherently non-constructive. We hope that our constructive proofs will interest also the logician who is not necessarily well-versed in HoTT.

All theorems in this paper have been mechanised in the Cubical Agda proof assistant. The proofs can be found here.

B. Related work

Our definition of CW complexes is based on the definition given by Buchholtz and Favonia in their work on cellular cohomology [5]. We develop the theory quite a bit further: we define cellular maps and cellular homotopies, and we prove their appurtenant approximation theorems. This lets us prove that cellular (co)homology is functorial without having to rely on a pre-existing (co)homology theory. This is especially valuable for the construction of cellular *homology*, as there is no pre-existing notion of homology that has been developed to the same extent as Eilenberg-MacLane cohomology.

Nevertheless, there is work by Graham on developing synthetic homology in HoTT using the Eilenberg-MacLane prespectrum [6]. The resulting functor is expected to satisfy the

Eilenberg-Steenrod axioms, but the additivity axiom remains an open question. Additionally, Christensen and Scoccola gave a proof of the Hurewicz theorem for this definition of homology [7]. We emphasise, however, that the proof given of the Hurewicz in this paper is vastly different in its approach and that it concerns a different homology theory than the one used by Christensen and Scoccola.

II. Background

In this section we will give a brief introduction to the elementary constructions and facts from HoTT which are used in this paper. We assume some level of familiarity with HoTT and refer the reader to the *HoTT Book* [1] whose notation we, for the most part, stay consistent with in this paper. Another excellent introduction is [8].

a) Π-types: for Π-types, we borrow `Agda` notation and often write $(a : A) \to B\,a$ instead of $\Pi_{x:a} B\,a$. Non-dependent Π-types are simply denoted $A \to B$. We may still use the traditional Π-notation when convenient.

b) Path types: given $x, y : A$, we write $x = y$ for their identity type. We refer to elements of this type as *paths*, and we write $\mathsf{refl}_x : x = x$ for the constant path. The *path induction* rule states that dependent functions $((y,p) : \Sigma_{y:A}(x = y)) \to B(y,p)$ are determined by their action on (x, refl_A).

c) Universes and pointed types: we write Type for the universe of types (at some implicit universe level) and $\mathsf{Type}_\star$ for the universe of pointed types, i.e. the type of pairs $(A, \star_A)$ where $A : \mathsf{Type}$ and $\star_A : A$. For simplicity, we generally write 'A is a pointed type' and leave the basepoint implicit. We always use the notation $\star_A$ for basepoints.

d) Pointed functions: given two pointed types A and B, the type of pointed functions $A \to_\star B$ is the type of pairs $(f, \star_f)$ where $f : A \to B$ is a function and $\star_f : f\,\star_A = \star_B$. We often simply write $f : A \to_\star B$ and leave $\star_f$ implicit.

e) Fibres, equivalences and univalence: we write $\mathsf{fib}_f(b)$ for the fibre of a function $f : A \to B$ over a point $b : B$. That is, $\mathsf{fib}_f(b) := \Sigma_{a:A}(f\,a = b)$. A function $f : A \to B$ whose fibres are contractible (i.e. pointed by a unique point) is called an equivalence. We write $f : A \simeq B$, and $f^{-1} : B \simeq A$ for the induced inverse. The identity $\mathsf{id} : A \to A$ is always an equivalence; the univalence axiom says precisely that the function $A = B \to A \simeq B$ defined by path induction by sending refl_A to the identity equivalence is itself an equivalence.

f) The Unit type and the empty type: we write $\mathbb{1}$ for the unit type, i.e. the inductive type with one unique constructor $\star_{\mathbb{1}} : \mathbb{1}$, and $\bot$ for the empty type.

A. Pushouts

Besides inductive types, we will also make heavy use of *higher inductive types* (HITs), which include path constructors in addition to point constructors. One of the arguably most important HITs in HoTT is the *pushout* of a span.

Definition 1 (Pushouts). *Given a span $Y \xleftarrow{f} X \xrightarrow{g} Z$, we define its pushout (as indicated in the diagram below) to be the HIT generated by point constructors $\mathsf{inl} : Y \to Y \sqcup^X Z$ and $\mathsf{inr} : Z \to Y \sqcup^X Z$, as well as a higher constructor $\mathsf{push} : (x : X) \to \mathsf{inl}\,(f\,x) = \mathsf{inr}\,(g\,x)$.*

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ Y & \to & Y \sqcup^X Z \end{array}$$

Given a span S, we may also write $\mathsf{PO}\,S$ for its pushout. We always take $Y \sqcup^X Z$ to be pointed by $\mathsf{inl}\,\star_Y$ (assuming Y is pointed). Pushouts will allow us to define most spaces of interest in this paper. The following three instances of pushouts are especially important for us.

a) Cofibres: we define the *cofibre* of a map $f : X \to Y$, denoted C_f, by $C_f := \mathbb{1} \sqcup^X Y$. To stay consistent with the existing literature, we write cfcod instead of $\mathsf{inr} : Y \to C_f$.

b) Wedge sums: given a dependent family of pointed types $A : I \to \mathsf{Type}_\star$, we define its *wedge sum*, denoted $\bigvee_{i:I}(A\,i)$, to be the cofibre of the obvious map $I \to \Sigma_{i:I}(A\,i)$. When we specifically wish to reason about binary wedge sums of two pointed types A and B, we may also define these by $A \vee B = A \sqcup^{\mathbb{1}} B$. We write $\iota_\vee$ for the canonical map $\bigvee_{i:I}(A\,i) \to \Pi_{i:I}(A\,i)$, which is definable whenever I has decidable equality.

c) Suspensions: we define the *suspension* of a type X, denoted ΣX, by $\Sigma X := \mathbb{1} \sqcup^A \mathbb{1}$. We use north and south to refer to $\mathsf{inl}\,\star$ and $\mathsf{inr}\,\star$ respectively, and $\mathsf{merid} : A \to \mathsf{north} = \mathsf{south}$ instead of push. Suspensions also allow us to define spheres inductively by setting $\mathbb{S}^{-1} := \bot$, i.e the empty type, and $\mathbb{S}^n := \Sigma \mathbb{S}^{n-1}$ for $n > -1$.

Let us state two elementary lemmas concerning pushouts which will be useful later. The following lemma is proved using standard pushout-pasting arguments.

Lemma 2. *Let $f : A \to B$ with A and B. We have*

1) $C_{(\mathsf{cfcod} : B \to C_f)} \simeq \Sigma A$
2) *$C_f \simeq \Sigma A \vee B$ if B is pointed and f is constant.*

We will also need the $3{\times}3$-*lemma* – an incredibly useful result which was first introduced in the HoTT literature by Brunerie [2, Lemma 1.8.3] whose notation we also borrow. Let A_{ij} be a commutative grid of types indexed by $I = \{0, 2, 4\}$ as in the diagram to the right. The $3{\times}3$-lemma says that *taking pushouts over rows and then columns is equivalent to taking pushouts over columns and then rows.* Let us unwrap this statement. Let $A_{\bullet i}$ and $A_{i \bullet}$ denote, respectively, the pushout along column i and the pushout along row i. That is, let $A_{\bullet i} := A_{0i} \sqcup^{A_{2i}} A_{4i}$ and $A_{i\bullet} := A_{i0} \sqcup^{A_{i2}} A_{i4}$. We produce a span $(A_{\bullet i})_{i \in I} := (A_{\bullet 0} \xleftarrow{f_{01} \sqcup^{f_{21}} f_{41}} A_{\bullet 2} \xrightarrow{f_{03} \sqcup^{f_{23}} f_{43}} A_{\bullet 4})$. We define $(A_{i\bullet})_{i\in I}$ similarly. Let $A_{\square\bullet} := \mathsf{PO}\,(A_{\bullet i})_{i \in I}$ and $A_{\bullet\square} := \mathsf{PO}\,(A_{\bullet i})_{i\in I}$.

$$\begin{array}{ccccc} A_{00} & \xleftarrow{f_{01}} & A_{02} & \xrightarrow{f_{03}} & A_{04} \\ {\scriptstyle f_{10}}\uparrow & & {\scriptstyle f_{12}}\uparrow & & \uparrow{\scriptstyle f_{14}} \\ A_{20} & \xleftarrow[f_{21}]{} & A_{22} & \xrightarrow{f_{23}} & A_{24} \\ {\scriptstyle f_{30}}\downarrow & & \downarrow{\scriptstyle f_{32}} & & \downarrow{\scriptstyle f_{34}} \\ A_{40} & \xleftarrow[f_{41}]{} & A_{42} & \xrightarrow[f_{43}]{} & A_{44} \end{array}$$

Lemma 3 (3×3-lemma). $A_{\square\bullet} \simeq A_{\bullet\square}$

B. Truncations

We say a type A is a (-2)-type if it is contractible (i.e. if it consists of a unique element) and, inductively, that it is an $(n+1)$-type if any identity type $x =_A y$ over A is a n-type. We refer to (-1)-types (i.e. types with at most one element) as *propositions* and 0-types (i.e. types which satisfy UIP) as *sets*. In HoTT, any type A can be turned into a n-type by forming its *n-truncation*, denoted $\|A\|_n$. This type is defined as a HIT with a point constructor $|-| : A \to \|A\|_n$ and a few additional constructors forcing $\|A\|_n$ to be an n-type. A detailed implementation can be found in [1, Section 7.3] but will not be needed here; all we shall need is the elimination property of the n-truncation which says that any (possibly dependent) function $f : (x : \|A\|_n) \to B\,x$ is uniquely determined by its action on canonical elements whenever B is a family of n-types. That is, the map $((x : \|A\|_n) \to B\,x) \to ((a : A) \to B\,|a|)$ is an equivalence. The philosophy of the elimination principle is that whenever we are trying to construct an element of a n-type, we may use $\|A\|_n$ and A interchangeably.

Truncations are crucial for internalising several notions and constructions from traditional mathematics in HoTT:

a) Choice: we say that a type A satisfies choice if the canonical map $\|(a : A) \to B\,a\|_{-1} \to ((a : A) \to \|B\,a\|_{-1})$ is an equivalence. A typical example of a type which satisfies choice is $\mathsf{Fin}(n) := \Sigma_{i:\mathbb{N}}(i < n)$, meaning that we do not need any axiom to get choice for families indexed over a finite set.

b) Existence: we define $\exists_{a:A}(B\,a) := \|\Sigma_{a:A}(B\,a)\|_{-1}$ to encode a notion of 'classical' or 'proof irrelevant' existence. If this type is inhabited, we say that there *merely* exists an element $a : A$ so that $B\,a$ holds.

c) Homotopy groups: given a pointed type A and an integer $n \geq 1$, we define the nth homotopy group of A by $\pi_n(A) := \|\mathbb{S}^n \to_\star A\|_0$. This type turns out to have a group structure, which is abelian for $n \geq 2$. The construction is functorial via post-composition; for a map $f : A \to_\star B$, we write $\pi_n(f) : \pi_n(A) \to \pi_n(B)$ for the functorial action. The construction is also invariant under n-truncation: the canonical map $\pi_n(A) \to \pi_n(\|A\|_n)$ is an isomorphism of groups.

d) Connectedness: we say that a type A is n-connected if $\|A\|_n$ is contractible. A function $f : A \to B$ is said to be n-connected if all of its fibres are. It is an easy fact that if f is n-connected, it induces an equivalence on truncations $\|A\|_n \simeq \|B\|_n$ (and thus also on π_n).

Another important fact about n-truncations is that they commute with path types, in the sense that for any $x, y : A$, the canonical map $\|x = y\|_n \to |x| =_{\|A\|_{n+1}} |y|$ is an equivalence [1, Theorem 7.3.12]. This principle, together with the elimination principle for $\mathbb{S}^n$, gives rise to the following elementary 'choice principle' for $\mathbb{S}^n$.

Lemma 4. *Given a dependent type $A : \mathbb{S}^n \to \mathsf{Type}$, there exists a function*

$$\mathsf{choose}_{\mathbb{S}^n} : ((x : \mathbb{S}^n) \to \|A\,x\|_{n-1}) \to \|(x : \mathbb{S}^n) \to A\,x\|_{-1}$$

III. CW Complexes in HoTT

A *CW complex* is a space which is constructed by an iterative process of attaching cells: start with a collection of points (0-dimensional cells), then connect some of them using 1-dimensional line segments to obtain a multigraph, then glue a collection of 2-dimensional discs to the multigraph, then 3-dimensional cells, and so on. This iterative construction is captured by the following definition in type theory:

Definition 5. *A **CW structure** is a sequence of types $(X_{-1} \xrightarrow{\iota_{-1}} X_0 \xrightarrow{\iota_0} X_1 \xrightarrow{\iota_1} \ldots)$ together with a **cardinality function** $c^X_{(-)} : \mathbb{N} \to \mathbb{N}$ and **attaching maps** $\alpha^X_i : \mathbb{S}^i \times \mathsf{Fin}(c^X_{i+1}) \to X_i$ satisfying the following two conditions.*

- (A1) *$X_{-1} \simeq \emptyset$,*
- (A2) *for each $i \geq -1$, the square to the right is a pushout.*

$$\begin{array}{ccc} S^i \times \mathsf{Fin}(c^X_{i+1}) & \xrightarrow{\mathsf{snd}} & \mathsf{Fin}(c^X_{i+1}) \\ {\scriptstyle \alpha^X_i}\downarrow & & \downarrow \\ X_i & \xrightarrow[\iota_i]{} & X_{i+1} \end{array}$$

The cardinality function indicates how many cells should be added at every stage, and the attaching maps α_i explain how the boundary of each $(i+1)$-dimensional cell is attached to the i-skeleton X_i. Finally, the pushout condition states that X_{i+1} is obtained from X_i by gluing cones along these boundaries, as in the 'hub and spokes' construction from [1, Section 6.7]. Since we will be using them a lot throughout the paper, we introduce a special notation for the pushout constructors of X_{i+1}: given $x : X_i$, $y : \mathsf{Fin}(c^X_{i+1})$ and $s : \mathbb{S}^i$, we write

- $\iota_i\,x : X_{i+1}$ (as indicated in the diagram),
- $\mathsf{cell}\,y : X_{i+1}$, and
- $\mathsf{glue}(s, y)$ for the path $\iota_i\,(\alpha^X_i\,(s, y)) = \mathsf{cell}\,y$.

We remark that unlike the usual definition from classical algebraic topology, our CW structures only allow a finite number of cells in each dimension. The reason for this limitation is that we are committed to being fully constructive, and we would quickly run into issues with the axiom of choice if we allowed arbitrary sets of cells. Nevertheless, our definition does allow infinite-dimensional CW structures with cells in every dimension. This provides a slight generalisation of the definition used by Buchholtz and Favonia [5], and allows us to encode important spaces, such as the infinite-dimensional projective planes. We will often denote CW structures simply by $X_* : \mathsf{CWstr}$ and leave ι_*, c^X_* and α^X_* implicit.

Definition 6. *Given a CW structure X_*, we define its **sequential colimit** to be the HIT X_∞ which consists of*

- *for every $x : X_n$, a point $[x]_n : X_\infty$,*
- *for every $x : X_n$, a path $\mathsf{push}\,x : [x]_n = [\iota_n\,x]_{n+1}$.*

We sometimes write $\iota_\infty\,x$ for $[x]_n$ when n is clear from context.

Definition 7. *We say that a type A is a **CW complex** if there merely exists some CW structure X_* such that A is the sequential colimit of X_*. Formally, we define*

$$\mathsf{CW} := \Sigma(A : \mathsf{Type})\,.\,\exists(X_* : \mathsf{CWstr})\,.\,X_\infty \simeq A.$$

In the rest of this paper, we will develop the theory of CW complexes, building up to a definition of cellular homology and a proof of the Hurewicz theorem. In doing so, we will take advantage of the fact that every CW complex is presented by a CW structure, which allows us to construct most properties and objects by induction on the dimension. Thus, our first endeavour shall be the development of a working theory of CW structures, starting with their natural notion of maps.

Definition 8. *A cellular map from X_* to Y_* is a pair (f_*, h_*) where $f_i : X_i \to Y_i$ and $h_i : (x : X_i) \to f_{i+1}(\iota_i\, x) = \iota_i(f_i\, x)$, as depicted on the diagram below:*

$$\begin{array}{ccccccc} X_{-1} & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & \dots \\ {\scriptstyle f_{-1}}\downarrow & {\scriptstyle h_0} & \downarrow{\scriptstyle f_0} & {\scriptstyle h_1} & \downarrow{\scriptstyle f_1} & & \\ Y_{-1} & \longrightarrow & Y_0 & \longrightarrow & Y_1 & \longrightarrow & \dots \end{array}$$

In simpler terms, a cellular map is a map which respects the dimensions, in the sense that it sends the n-dimensional skeleton of the source to the n-dimensional skeleton of the target. For simplicity, we generally write $f_* : X_* \to Y_*$ for a cellular map, leaving h_* implicit. Every cellular map from X_* to Y_* gives rise to a function f_∞ between their colimits:

$$\begin{aligned} & f_\infty : X_\infty \to Y_\infty \\ & f_\infty\, [x]_n && := \quad [f_n\, x]_n \\ & \mathsf{ap}_{f_\infty}(\mathsf{push}\, x) && := \quad \mathsf{push}\,(f_n\, x) \cdot \mathsf{ap}_{[-]_{n+1}}\,(h_n\, x). \end{aligned}$$

The identity can be presented as a cellular map, and the obvious composition of two cellular maps yields the composition of the colimits. Furthermore, this composition operation is associative and unital. All this data assembles into a category CWstr whose objects are CW structures, and whose mapping sets are given by (the set-truncation of) cellular maps. The colimit operation then defines a functor from CWstr to the category CW of CW complexes and ordinary maps.

The interplay between CW and CWstr will be a recurring theme of this paper: our main object of interest is the category CW, but we find that it does not offer sufficient control over the objects and the morphisms. Instead, we define all of our constructions in CWstr, taking advantage of the inductive description of spaces and maps, before transporting them to CW. Our main tool for this transport step will be the *cellular approximation theorem*, which provides a partial inverse to the colimit functor.

A. Pushouts of CW structures

Before embarking on the proof of the cellular approximation theorem, it might be good to look at a concrete example of a CW structure, to help the reader build intuition. For this purpose, we shall explain how to construct the homotopy pushout of two cellular maps. This construction will play an important role later down the line, as the definition of a homology theory requires our category of CW complexes to be equipped with pushouts.

Definition 9. *Let X_*, Y_*, Z_* be three CW structures, and $(f_*, h_*) : X_* \to Y_*$ and $(g_*, k_*) : X_* \to Z_*$ be two cellular maps. We define the pushout of the span $Y_* \xleftarrow{f_*} X_* \xrightarrow{g_*} Z_*$ to be the CW structure $(Y \sqcup^X Z)_*$ defined by letting $(Y \sqcup^X Z)_i$ be the pushout $Y_i \sqcup^{X_{i-1}} Z_i$, i.e. the pushout of the span $Y_i \xleftarrow{\iota_{i-1} \circ f_{i-1}} X_{i-1} \xrightarrow{\iota_{i-1} \circ g_{i-1}} Z_i$.*

Inclusions: *The inclusions $(Y \sqcup^X Z)_i \to (Y \sqcup^X Z)_{i+1}$ are the obvious maps induced by the corresponding inclusions for X_*, Y_* and Z_*.*
Cells: *We define the cell cardinalities $c_*^{Y \sqcup^X Z}$ in terms of those of X_*, Y_* and Z_* by letting $c_i^{Y \sqcup^X Z} = c_i^Y + c_i^Z + c_{i-1}^X$.*
Attaching maps: *Finally, we define the attaching maps as*

$$\begin{aligned} \alpha_i^{Y \sqcup^X Z} \;&:\; \textstyle\sum_{c \in \{c_{i+1}^Y, c_{i+1}^Z, c_i^X\}} \mathbb{S}^i \times \mathsf{Fin}(c) \;\to\; Y_i \sqcup^{X_{i-1}} Z_i \\ \alpha_i^{Y \sqcup^X Z} \;&:=\; \upsilon_i + \zeta_i + \chi_i \end{aligned}$$

where we define $\upsilon_i := \mathsf{inl} \circ \alpha_i^Y$, $\zeta_i := \mathsf{inr} \circ \alpha_i^Z$, and $\chi_i : \mathbb{S}^i \times \mathsf{Fin}(c_i^X) \to P_i$ is defined by $\mathbb{S}^i$-induction: on point constructors by setting $\chi_i(\mathsf{north}, y) := \mathsf{inl}\,(f_{i+1}(\mathsf{cell}\, y))$ and $\chi_i(\mathsf{south}, y) := \mathsf{inr}\,(g_{i+1}(\mathsf{cell}\, y))$, and on the path constructor by letting $\mathsf{ap}_{\chi_i(-,y)}(\mathsf{merid}\ x)$ be the composite path

$$\mathsf{inl}\,(\dots) \xrightarrow{\mathsf{ap}_{\mathsf{inl}}\, l} \mathsf{inl}\,(\dots) \xrightarrow{\mathsf{push}\,(\alpha_{i-1}^X(x,y))} \mathsf{inr}\,(\dots) \xrightarrow{\mathsf{ap}_{\mathsf{inr}}\, r^{-1}} \mathsf{inr}\,(\dots)$$

where $l : f_i(\mathsf{cell}\, y) = \iota_{i-1}(f_{i-1}(\alpha_{i-1}^X(x, y)))$ and is defined by $l := \mathsf{ap}_{f_i}(\mathsf{glue}(x, y)^{-1}) \cdot h_{i-1}(\alpha_{i-1}^X(x, y))$, and similarly for $r : g_i(\mathsf{cell}\, y) = \iota_{i-1}(g_{i-1}(\alpha_{i-1}^X(x, y)))$.

Proposition 10. *Definition 9 satisfies (A1) and (A2).*

Proof. The proof mostly follows from the 3×3 lemma. The interested reader can consult the formalised version. ☐

Note that the colimit of this definition is the expected pushout:

$$\operatorname*{colim}_{i \to \infty} \big(Y \sqcup^X Z\big)_i = \operatorname*{colim}_{i \to \infty} \big(Y_i \sqcup^{X_{i-1}} Z_i\big) \simeq Y_\infty \sqcup^{X_\infty} Z_\infty$$

and thus Definition 9 does indeed provide a CW structure for the pushout of a span in CWstr. Now, if we want to extend this construction to the category CW, we need to work with arbitrary maps between the colimits instead of cellular maps. This is one out of a handful places where cellular approximation is needed.

B. Finite structures and the cellular approximation theorem

In classical algebraic topology, the cellular approximation theorem is a cornerstone result which states that any continuous function between two CW complexes is homotopic to a cellular map. This seems perfect for, for instance, extending our constructions of pushouts to CW, but unfortunately this theorem appears to be out of reach in our constructive framework: the standard proof involves considerations of point-set topology and the use of the axiom of choice. However, what we can prove is a *synthetic* and *finitary* version of the theorem, which informally states that the cellular approximation theorem holds when the domain has finitely many cells. This will be the main result of this subsection.

Before providing the precise statement for our constructive cellular approximation theorem, let us start with a brief digression about finite subcomplexes and substructures – this will allow us to formulate a statement that is somewhat more flexible than the one suggested above. We say that a CW structure is *finite* (of dimension n) if the maps in its underlying sequence of types are equivalences starting from dimension n.

Given any CW structure X_*, there is a canonical way to restrict it to a finite CW structure $X_*^{(n)}$ with the following definitions:

$$X_i^{(n)} := \begin{cases} X_i & \text{if } i < n \\ X_n & \text{otherwise} \end{cases} \qquad c_i^{(n)} := \begin{cases} c_i & \text{if } i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The structure $X_*^{(n)}$ trivially satisfies $X_\infty^{(n)} \simeq X_n$. We will use the same notation for cellular maps and cellular homotopies, writing $f_*^{(n)}$ and $p_*^{(n)}$ respectively for the restrictions of a cellular map f_* and a cellular homotopy p_* to the n-skeleton of the domain. For ease of notation, we also define $X_*^{(\infty)} := X_*$ (and similarly for $f_*^{(\infty)}$ and $h_*^{(\infty)}$).

Definition 11. *Let X_* and Y_* be two CW structures, and let $f : X_\infty \to Y_\infty$ be an arbitrary map between their colimits. A **cellular n-approximation** of f is the data of a cellular map $(f_*, h_*) : X_*^{(n)} \to Y_*$ along with a homotopy*

$$t : (x : X_n) \to f(\iota_\infty x) = \iota_\infty(f_n\, x).$$

Our first cellular approximation states that n-approximations always exist for n finite. To get there, we will need the help of two easy lemmas.

Lemma 12. *For any CW structure X_*, the inclusion map $\iota_i : X_i \to X_{i+1}$ is $(i-1)$-connected.*

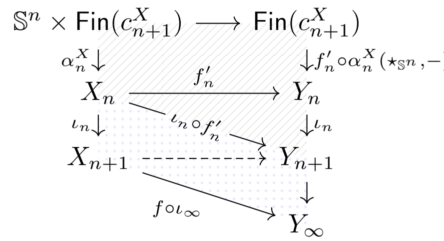
Proof. It is a general fact that given any span $B \xleftarrow{f} A \xrightarrow{g} C$, the map $\mathsf{inl} : B \to B \sqcup^A C$ is as connected as g [2, Proposition 2.3.10]. In our case, X_{i+1} is defined as the pushout of the span $X_i \leftarrow \mathbb{S}^i \times \mathsf{Fin}(c_{i+1}) \xrightarrow{\mathsf{snd}} \mathsf{Fin}(c_{i+1})$, and thus it suffices to show that the projection $\mathsf{snd} : \mathbb{S}^i \times \mathsf{Fin}(c_{i+1}) \to \mathsf{Fin}(c_{i+1})$ is $(i-1)$-connected. This is evident as its fibres are equivalent to $\mathbb{S}^i$, which is $(i-1)$-connected. ☐

Lemma 13. *For any CW structure X_*, the inclusion map $\iota_\infty : X_i \to X_\infty$ is $(i-1)$-connected.*

Proof. It follows immediately from Lemma 12 that all of the maps $X_{i+k} \xrightarrow{\iota_{i+k}} X_{i+k+1}$ are at least $(i-1)$ connected. As a consequence, their transfinite composition $\iota_\infty : X_i \to X_\infty$ is also $(i-1)$-connected [9, Corollary 7.7]. ☐

Theorem 14 (First cellular approximation theorem)**.** *Let X_* and Y_* be CW structures. For any map $f : X_\infty \to Y_\infty$ and $n : \mathbb{N}$, there merely exists a cellular n-approximation of f.*

Proof. The proof proceeds by induction on n. The base case, $n = -1$, is trivial. For the inductive step, assume that we have an n-approximation f'_* of f. We will use f'_* to *merely* construct an $(n+1)$-approximation $f_* : X_*^{(n+1)} \to Y_*$ (the fact that we are only aiming for mere existence allows us to use the elimination rule for propositional truncations a finite number of times). We define $f_i := f'_i$ for all $i \leq n$. It remains to define f_{n+1} and its associated homotopies. Consider the following (not necessarily commutative) diagram:



If we can construct f_{n+1} as the dashed map above in a way that makes all triangles commute, we are done. By the elimination principle of pushouts, the dashed map exists if we can fill the shaded area. In other words we need construct an element of type $\|((x, y) : \mathbb{S}^n \times \mathsf{Fin}(c_{n+1})) \to F(x, y) = F(\star, y)\|_{-1}$ for $F := \iota_n \circ f'_n \circ \alpha^X_n$. Using finite choice and Lemma 4, this corresponds to constructing, for every $y : \mathsf{Fin}(c_{n+1})$, a family of paths $(x : \mathbb{S}^n) \to \|F(x, y) = F(\star, y)\|_{n-1}$. By Lemma 13, the map $\iota_\infty : Y_{n+1} \to Y_\infty$ is n-connected and therefore its action on path spaces, $\mathsf{ap}_{\iota_\infty}$, is $(n-1)$-connected. Thus $\|F(x, y) = F(\star, y)\|_{n-1}$ is equivalent to $\|\iota_\infty(F(x, y)) = \iota_\infty(F(\star, y))\|_{n-1}$. Since the dotted area of the diagram commutes, it suffices to show that the outermost diagram commutes, which is a consequence of the homotopies associated with f'_*. Therefore, the dashed map exists. The remaining homotopy involved in the definition of a cellular approximation holds by construction. ☐

Corollary 15. *For any span of CW complexes $Y \xleftarrow{f} X \xrightarrow{g} Z$ with X finite, the pushout $Y \sqcup^X Z$ is a CW complex.*

Unfortunately, our finitary approximation theorem is not quite strong enough to prove the existence of all pushouts in CW. One option to remedy this would be to assume the axiom of countable choice, which allow us to deduce the mere existence of an ∞-approximation from the mere existence of a n-approximation for every n. This would, however, limit the generality of our theorems (countable choice does not hold in arbitrary infinity toposes), so we will refrain from doing so.

Question 16. *Can we prove that every map between CW complexes merely has an ∞-approximation without using the axiom of countable choice?*

Since we do not know the answer to this question, we will have to work with finite cellular approximations for the rest of this paper. For this purpose, we introduce a category $\mathsf{CW}^{(n)}$ of n-truncated CW complexes.

Definition 17. *A n-truncated CW complex is a n-truncated type A for which there merely exists a CW structure X_* of dimension $n+1$ such that $A \simeq \|X_{n+1}\|_n$.*

Note the mismatch between the dimension of the structure and the truncation level. This mismatch is here so that we may define a truncation functor trunc_n from CW to $\mathsf{CW}^{(n)}$: we can send the pair $(A, |X_*|)$ to $(\|A\|_n, |X_*^{(n+1)}|)$, and the isomorphism condition holds because $\|X_\infty\|_n \simeq \|X_{n+1}\|_n$.

We also introduce a corresponding category $\mathsf{CWstr}^{(n)}$ whose objects are CW structures of dimension $n+1$, and whose morphisms are cellular maps of dimension n.

C. Cellular homotopies and the 2nd approximation theorem

In essence, the first cellular approximation tells us that there *merely* exists an inverse to the colimit operation for finite cellular maps. This already lets us transfer some constructions from CWstr to CW, but we would ideally like to get rid of that propositional truncation and define a proper approximation functor from CW to CWstr – or at least from $\mathsf{CW}^{(n)}$ to $\mathsf{CWstr}^{(n)}$, to avoid choice issues. Unfortunately, this turns out to be problematic, as the cellular approximation theorem is inconsistent without the propositional truncation.

Theorem 18. *The set-truncated version of Theorem 14 is false.*

Proof. Both $\mathbb{1}$ and $\mathbb{S}^1$ can be presented by finite CW structures (which we will denote by $\mathbb{1}_*$ and ${\mathbb{S}^1}_*$), with only a 0-cell for the former and a 0-cell plus a 1-cell for the latter. Given any $x : \mathbb{S}^1$, define $\widehat{x} : \mathbb{1} \to \mathbb{S}^1$ to be the corresponding function. A cellular approximation of $\widehat{x}$ means that we can factors it as $\mathbb{1} \xrightarrow{\sim} \mathbb{1}_0 \xrightarrow{\widehat{x}_0} {\mathbb{S}^1}_0 \to \mathbb{S}^1$, which in turn implies that x is equal to the basepoint $\star_{\mathbb{S}^1}$. Therefore, the set-truncated version of Theorem 14 provides a proof of $(x : \mathbb{S}^1) \to \|x = \star_{\mathbb{S}^1}\|_0$. By Lemma 4, this entails $\|(x : \mathbb{S}^1) \to x = \star_{\mathbb{S}^1}\|_{-1}$ which by truncation elimination implies that $\mathbb{S}^1$ is contractible. But this is provably false in HoTT [10]. □

This problem ultimately stems from a mismatch between the notion of equality for morphisms in CW and the notion of equality for morphisms in CWstr. Since any homotopy gives rise to an equality between maps, the morphisms in CW should be understood as maps *up to homotopy*, while the equality between morphisms in CWstr is much closer in spirit to a strict equality. Therefore, if we want to frame our approximation theorem as functor, we need to quotient the morphisms of the target category by an adequate notion of homotopy.

Definition 19. *A* ***cellular homotopy*** *between cellular maps $f_*, g_* : X_* \to Y_*$ is a family*

$$p_i : (x : X_i) \to \iota_i(f_i(x)) =_{Y_{i+1}} \iota_i(g_i(x))$$

with fillers $q_i\, x$, for each $i > 0$ and $x : X_i$, of the following square.

$$\begin{array}{ccc} \iota_{i+1}(f_{i+1}(\iota_i\, x)) & \xrightarrow{p_{i+1}(\iota_i\, x)} & \iota_{i+1}(g_{i+1}(\iota_i\, x)) \\ \uparrow & q_i\, x & \uparrow \\ \iota_{i+1}(\iota_i\,(f_i\, x)) & \xrightarrow[\mathsf{ap}_{\iota_{i+1}}(p_i\, x)]{} & \iota_{i+1}(\iota_i\,(g_i\, x)) \end{array}$$

We use the notation $(p_*, q_*) : f_* \sim g_*$ or simply $p_* : f_* \sim g_*$ when the q_i's are clear from context.

One can easily prove that composition of cellular maps is invariant with respect to cellular homotopy. This lets us define the category Ho(CWstr), whose objects are CW structures and whose morphisms are cellular maps up to cellular homotopy. Furthermore, the existence of a cellular homotopy between f_* and g_* implies that their colimits are homotopic, or in other words, that $f_\infty = g_\infty$. This means that the colimit functor factors through Ho(CWstr).

That new colimit functor *almost* induces an equivalence between the categories $\mathsf{CW}^{(n)}$ and $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$. In order to prove this, we will need to extend our approximation theorem to cellular homotopies. Because the caveats regarding the axiom of countable choice still apply, we start by introducing a notion of finite approximation for cellular homotopies.

Definition 20. *Let $f_*, g_* : X_* \to Y_*$ be two cellular maps, and let $p : (x : X_\infty) \to f_\infty(x) = g_\infty(x)$ be a homotopy between their colimits. A* ***cellular*** n***-approximation*** *of p is a cellular homotopy $p_* : f_*^{(n)} \sim g_*^{(n)}$ equipped with a filler of the following square for each $x : X_n$.*

$$\begin{array}{ccc} \iota_\infty(\iota_n(f_n(x))) & \xrightarrow{\mathsf{ap}_{\iota_\infty}(p_n(x))} & \iota_\infty(\iota_n(g_n(x))) \\ \uparrow & & \uparrow \\ f_\infty(\iota_\infty\, x) & \xrightarrow[p(\iota_\infty\, x)]{} & g_\infty(\iota_\infty\, x) \end{array}$$

We are now ready to state the second cellular approximation theorem. Its proof follows the same strategy as Theorem 14, so we omit it and refer the reader to the computer formalisation.

Theorem 21 (Second cellular approximation theorem)**.** *Let $f_*, g_* : X_* \to Y_\infty$ be cellular maps and $p : f_\infty \sim g_\infty$. For any $n : \mathbb{N}$, there merely exists an n-approximation of p.*

Theorem 21 implies that taking the colimit of a cellular map between two CW structures in $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$ is an injective operation. On the other hand, Theorem 14 implies that it is a surjective operation. Therefore, the colimit induces a fully faithful functor from $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$ to $\mathsf{CW}^{(n)}$. Since this functor is essentially surjective in the sense of [1, Chapter 9], we get the following result as a corollary.

Corollary 22. *The colimit functor induces a weak equivalence between $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$ and $\mathsf{CW}^{(n)}$. Equivalently, $\mathsf{CW}^{(n)}$ is the Rezk completion of $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$.*

$$\begin{array}{ccccc} \mathsf{CW} & \xleftarrow{\mathsf{colim}} & \mathsf{Ho}(\mathsf{CWstr}) & \longleftarrow & \mathsf{CWstr} \\ \downarrow{\scriptstyle \mathsf{trunc}_n} & & \downarrow{\scriptstyle \mathsf{trunc}_n} & & \downarrow{\scriptstyle \mathsf{trunc}_n} \\ \mathsf{CW}^{(n)} & \xleftarrow[\mathsf{colim}]{\sim} & \mathsf{Ho}(\mathsf{CWstr}^{(n)}) & \longleftarrow & \mathsf{CWstr}^{(n)} \end{array}$$

Fig. 1. The categories at play

The relations between the various categories defined so far are summarised in Figure 1. This diagram gives us a systematic way of lifting a functor F defined over CWstr to a functor defined over CW: first, if the functor F happens to use only a finite number of dimensions, it can be factored as $\overline{F} \circ \mathsf{trunc}_n$ for some functor $\overline{F}$ defined over $\mathsf{CWstr}^{(n)}$. Then, if we manage to prove that $\overline{F}$ is invariant under cellular homotopy, we can extend it to a functor $\widetilde{F}$ defined over $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$. Finally, if the target is a univalent category, $\widetilde{F}$ can be extended to a functor defined over the Rezk completion of $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$, which is $\mathsf{CW}^{(n)}$. By composing the result with the truncation functor, we get a lift of F to CW.

IV. Cellular Homology

In algebraic topology, the homology groups of a space is a family of topological invariants which are somewhat similar to homotopy groups, in that they intuitively measure the number of n-dimensional holes, but are much simpler to compute. There is a plethora of *homology theories* (roughly, different definitions for these homology groups) but among them, one is especially relevant to our interests: the theory of *cellular homology* is defined in terms of their CW structures, and is particularly well suited for computation. Developing cellular homology in HoTT gives a new meaning to the adjective *computational*. Through the Curry-Howard correspondence, it provides formally verified computations of homology groups, facilitating the idea of 'proof by computation' – a central idea in computer formalisation of synthetic homotopy theory [2], [11], [12], [13].

In this section, we define a (reduced) homology functor $\widetilde{H}_i^{\mathsf{str}} : \mathsf{CWstr} \to \mathsf{AbGrp}$ which we then lift to a functor $\widetilde{H}_i^{\mathsf{cw}}$ over CW using our freshly proved cellular approximation theorem. This provides the first complete definition of cellular homology in HoTT. We also state the Eilenberg-Steenrod axioms and prove that our functors satisfy them, thereby showing that they deserve the name of a homology theory.

A. A crash course in homological algebra

The first step in the definition of homology groups is to approximate CW structures by *cellular chain complexes*. Before doing so, however, we need some preliminary background on chain complexes, as well as a definition of the homology groups of a chain complex.

Definition 23. *A* ***chain complex*** *is a sequence of abelian groups (called i-chains)*

$$\dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} \dots$$

where the maps ∂_i (called boundary maps) are group homomorphisms satisfying the equation $\partial_i \circ \partial_{i+1} = 0$.

Definition 24. *A* ***chain map*** *$\phi_* : C_* \to D_*$ is a collection of group homomorphisms $\phi_i : C_i \to D_i$ compatible with boundary maps in the sense that $\phi_i \circ \partial^C_{i+1} = \partial^D_{i+1} \circ \phi_{i+1}$.*

There are natural definitions of chain map composition (levelwise composition) and of the identity chain map (the levelwise identity). This lets us define the category Ch whose objects are chain complexes and whose morphisms are chain maps. We also have a natural notion of chain homotopy.

Definition 25. *A* ***chain homotopy*** *η_* between two chain maps $\phi_*, \psi_* : C_* \to D_*$ is a sequence of group homomorphisms $\eta_i : C_i \to D_{i+1}$ such that $\phi_i - \psi_i = \partial^D_{i+1} \circ \eta_i + \eta_{i-1} \circ \partial^C_i$.*

Chain homotopies are compatible with composition, which lets us define the homotopy category of chain complexes $\mathsf{Ho}(\mathsf{Ch})$ whose morphisms are chain maps up to chain homotopy. We finally arrive at the definition of homology groups, which is the natural analogue of homotopy groups in the category of chain complexes.

Definition 26 (Homology groups)**.** *We define the nth homology group of a chain complex (C_*, ∂_*) by $H_n(C_*) := \ker \partial_n / \operatorname{Im} \partial_{n+1}$.*

We remark that the quotient in the definition above is well-defined since the boundary equation $\partial_i \circ \partial_{i+1} = 0$ ensures that $\operatorname{Im} \partial_{i+1} \subseteq \ker \partial_i$. Furthermore, any chain map $\phi_* : C_* \to D_*$ induces a homomorphism $H_n(\phi_*) : H_n(C_*) \to H_n(D_*)$, and furthermore does so in a functorial way. Thus, the definition of the nth homology group can be presented as a functor from Ch to the category of abelian groups AbGrp. Lastly, a standard argument shows that the existence of a chain homotopy between two chain maps ϕ_* and ψ_* implies that $H_n(\phi_*) \cong H_n(\psi_*)$. Therefore, the definition of H_n factors through the category $\mathsf{Ho}(\mathsf{Ch})$. This concludes our definition of the homology groups of a chain complex.

B. Sphere bouquets and reduced cellular homology

We are now in position to define the cellular chain complex associated to a CW structure X_*. The definition for the abelian groups of n-chains is rather straightforward:

- when $n \geq 0$, we set $C_n := \mathbb{Z}[c^X_n]$, i.e. C_n is the free abelian group with a generator for each n-cell in X_*,
- when $n = -1$, we set $C_{-1} := \mathbb{Z}$ (in technical terms, this means that we are defining the *augmented* chain complex of X_*, but we will not go into detail here),
- when $n < -1$, we define C_n to be the trivial group.

The definition of the boundary maps is slightly more involved. In positive degrees, our goal is to construct a homomorphism of free abelian groups $\partial_{i+1} : \mathsf{Hom}(\mathbb{Z}[c^X_{i+1}], \mathbb{Z}[c^X_i])$. To do so, we will exploit the fact that free abelian groups are closely related to wedge sums of spheres, which we call *sphere bouquets*. This approach is essentially a reinterpretation of the definition used by May [14] and Buchholtz and Favonia [5]. In what follows, and for the remainder of the paper, we will use somewhat non-standard terminology and say that a type A is finite if $A \simeq \mathsf{Fin}(k)$ for some k.

Definition 27. *Given a finite type A and an integer $n \geq 0$, define the* ***sphere bouquet*** *of cardinality $|A|$ and dimension n to be the type $\bigvee_A \mathbb{S}^n$, i.e. the wedge sum of $|A|$ n-spheres.*

Before clarifying the relation between sphere bouquets and free abelian groups, we first need to recall some well-known facts about the *degree* of an endo-function of $\mathbb{S}^n$. For any $n > 0$, there is an isomorphism deg from $\pi_n(\mathbb{S}^n)$ to $\mathbb{Z}$. In fact, this definition extends to any (not necessarily pointed) map $f : \mathbb{S}^n \to \mathbb{S}^n$. This is done by noting that the 'forgetful map' $\|\mathsf{fst}\|_0 : \pi_n(\mathbb{S}^n) \to \|\mathbb{S}^n \to \mathbb{S}^n\|_0$ is an equivalence. This allows us to define a degree map by the composition

$$(\mathbb{S}^n \to \mathbb{S}^n) \xrightarrow{|-|} \|\mathbb{S}^n \to \mathbb{S}^n\|_0 \xrightarrow{\|\mathsf{fst}\|_0^{-1}} \|\mathbb{S}^n \to_\star \mathbb{S}^n\|_0 \xrightarrow{\mathsf{deg}} \mathbb{Z}$$

We allow some overloading of notation by also using deg to denote the above composition. In addition to inducing an isomorphism of groups, deg has a few useful properties.

Proposition 28. *The degree map commutes with suspensions, i.e. any* $f : \mathbb{S}^n \to \mathbb{S}^n$ *is of the same degree as its suspension* $\Sigma f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$. *Additionally,* deg *takes function composition to integer multiplication, i.e.* $\mathsf{deg}(f \circ g) = \mathsf{deg}\, f \cdot \mathsf{deg}\, g$.

As the degree map has been well-studied in HoTT already [5], [15], we omit the proof of Proposition 28. This degree function has a natural generalisation to sphere bouquets, which we call the *bouquet degree* function (bdeg). It is defined by the following composition of arrows:

$$\begin{array}{ccc} (\bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n) & & \Pi_A \Pi_B (\mathbb{S}^n \to \mathbb{S}^n) \\ \downarrow & & \downarrow (\mathsf{deg}^*)^* \\ (\Pi_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n) & & \Pi_A \Pi_B \mathbb{Z} \\ \iota_\vee{}^* \downarrow & & \downarrow \wr \\ (\Pi_A \mathbb{S}^n \to \Pi_B \mathbb{S}^n) & \longrightarrow & \mathsf{Hom}(\mathbb{Z}[A], \mathbb{Z}[B]) \end{array}$$

where the last two equivalences are defined using the universal property of the free abelian group. The bouquet degree map immediately inherits properties corresponding to those listed in Proposition 28:

Proposition 29. *Let* A, B *and* C *be finite types,* $n \geq 0$. *The following facts hold.*

1) *The bouquet degree function induces a group homomorphism* $\left\|\bigvee_A \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}\right\|_0 \to \mathsf{Hom}(\mathbb{Z}[A], \mathbb{Z}[B])$, *where the group structure on the left hand side is the natural extension of the group structure on* $\pi_{n+1}(\mathbb{S}^{n+1})$.
2) *The bouquet degree function commutes with suspension, i.e. any* $f : (\bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n)$ *is of the same degree as its suspension* $\Sigma f : (\Sigma(\bigvee_A \mathbb{S}^n) \to \Sigma(\bigvee_B \mathbb{S}^n))$, *where the bouquet degree of the latter function is well-defined since* $\Sigma(\bigvee_X \mathbb{S}^n) \simeq \bigvee_X \mathbb{S}^{n+1}$ *for any* X.
3) *The bouquet degree function respects composition, i.e. for* $f : \bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$ *and* $g : \bigvee_B \mathbb{S}^n \to \bigvee_C \mathbb{S}^n$ *we have* $\mathsf{bdeg}(g \circ f) = \mathsf{bdeg}\, g \circ \mathsf{bdeg}\, f$.

We can now return to the construction of the boundary maps: we would like to define, for any CW structure X_*, a homomorphism $\partial_{i+1} : \mathbb{Z}[c^X_{i+1}] \to \mathbb{Z}[c^X_i]$. By applying our bouquet degree function, it suffices to construct a function $d_i : \bigvee_{\mathsf{Fin}(c^X_{i+1})} \mathbb{S}^{i+1} \to \bigvee_{\mathsf{Fin}(c^X_i)} \mathbb{S}^{i+1}$. We recall from [5] that there is an equivalence $e : X_i / X_{i-1} \simeq \bigvee_{\mathsf{Fin}(c^X_i)} \mathbb{S}^i$. When $i > 0$, we obtain it by considering the following diagram.

$$\begin{array}{ccccc} \mathbb{S}^{i-1} \times \mathsf{Fin}(c^X_i) & \xrightarrow[\alpha^X_{i-1}]{} & X_{i-1} & \longrightarrow & \mathbb{1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Fin}(c^X_i) & \longrightarrow & X_i & \longrightarrow & \Sigma(\bigvee_{\mathsf{Fin}(c^X_i)} \mathbb{S}^{i-1}) \end{array}$$

Since the outermost square is a pushout, we know, by pushout pasting, that so is the right square. The construction of e is completed by observing that suspension commutes with wedge sums. When $i = 0$, the equivalence is obtained by noting that $X_0 / X_{-1} \simeq X_0 + \mathbb{1}$ which allows us to identify the appended point with the basepoint in $\bigvee_{\mathsf{Fin}(c^X_0)} \mathbb{S}^0$. We may now construct the desired map d_{i+1} by considering the composition

$$\bigvee_{\mathsf{Fin}(c^X_{i+1})} \mathbb{S}^{i+1} \xrightarrow{\sim} X_{i+1}/X_i \xrightarrow{\mathsf{pinch}} \Sigma X_i \xrightarrow{\Sigma\mathsf{cfcod}} \Sigma(X_i/X_{i-1}) \xrightarrow{\sim} \bigvee_{\mathsf{Fin}(c^X_i)} \mathbb{S}^{i+1}$$

and finally, we set $\partial_{i+1} = \mathsf{bdeg}\, d_{i+1}$. Note that this whole construction is only valid for $i > -1$. To complete the definition, we define $\partial_0 : \mathbb{Z}[c^X_0] \to \mathbb{Z}$ by sending every generator of $\mathbb{Z}[c^X_0]$ to 1, and lastly we let the maps in negative dimension be trivial.

Proposition 30. *The boundary maps satisfy* $\partial_i \circ \partial_{i+1} = 0$.

Proof. First, assume that $i > 0$. We compute:

$$\begin{aligned} \partial_i \circ \partial_{i+1} = \mathsf{bdeg}\, d_i \circ \mathsf{bdeg} d_{i+1} &= \mathsf{bdeg}\,(\Sigma d_i) \circ \mathsf{bdeg}\,(d_{i+1}) \\ &= \mathsf{bdeg}(\Sigma d_i \circ d_{i+1}) \end{aligned}$$

We are done if we can show that $\Sigma d_i \circ d_{i+1} = 0$. This composition of maps is defined as follows.

$$\begin{array}{l} \bigvee_{c^X_{i+1}} \mathbb{S}^{i+1} \to X_{i+1}/X_i \longrightarrow \Sigma X_i \dashrightarrow \Sigma(X_i/X_{i-1}) \dashrightarrow \bigvee_{c^X_i} \mathbb{S}^{i+1} \\ \dashrightarrow \Sigma \bigvee_{c^X_i} \mathbb{S}^i \dashrightarrow \Sigma(X_i/X_{i-1}) \dashrightarrow \Sigma^2 X_{i-1} \to \Sigma^2(X_{i-1}/X_{i-2}) \to \Sigma \bigvee_{c^X_{i-1}} \mathbb{S}^i \end{array}$$

It is enough to show that the dashed composition $\Sigma X_i \to \Sigma^2 X_{i-1}$ is trivial. By tracing the construction of the maps involved, it is easy to see that the map is given by

$$\Sigma X_i \xrightarrow{\Sigma\mathsf{cfcod}} \Sigma X_i / X_{i-1} \xrightarrow{\Sigma\mathsf{pinch}} \Sigma^2 X_{i-1}$$

which is equal to functorial action of Σ on $\mathsf{pinch} \circ \mathsf{cfcod} : X_i \to \Sigma X_{i-1}$. This is constant by definition. The case $i = 0$ follows by an explicit computation of the maps ∂_1 and ∂_0. □

At this point, we have a proper definition for the cellular chain complex of a CW structure. It remains to show that this construction lifts to a functor from CWstr to Ch.

Let $f_* : X_* \to Y_*$ be a cellular map. Because f_* is cellular, it determines a map $X_i/X_{i-1} \to Y_i/Y_{i-1}$. With a bit of help from the equivalence e that we defined earlier, we can define a map of sphere bouquets $\widetilde{f_i}$ as follows:

$$\bigvee_{\mathsf{Fin}(c^X_i)} \mathbb{S}^i \xrightarrow{e^{-1}} X_i/X_{i-1} \xrightarrow{f_i/f_{i-1}} Y_i/Y_{i-1} \xrightarrow{e} \bigvee_{\mathsf{Fin}(c^Y_i)} \mathbb{S}^i.$$

We may thus define the functorial action of f on i-chains, $\overline{f}_i : \mathsf{Hom}(\mathbb{Z}[c^X_i], \mathbb{Z}[c^Y_i])$, by setting $\overline{f}_i = \mathsf{bdeg}\, \widetilde{f_i}$. Let us verify that it is a chain map, i.e. that $\partial_{i+1} \circ \overline{f}_{i+1} = \overline{f}_i \circ \partial_{i+1}$. Using the fact that bdeg respects suspension and composition, this is equivalent to $\mathsf{bdeg}\,(d_{i+1} \circ \widetilde{f}_{i+1}) = \mathsf{bdeg}(\Sigma \widetilde{f_i} \circ d_{i+1})$. Let us simply show that $d_{i+1} \circ \widetilde{f}_{i+1} = \Sigma \widetilde{f_i} \circ d_{i+1}$. That is, we will show that the outer square commutes in the diagram below:

$$\begin{array}{ccccccccc} \bigvee_{c^X_{i+1}} \mathbb{S}^{i+1} & \to & X_{i+1}/X_i & \to & \Sigma X_i & \to & \Sigma(X_i/X_{i-1}) & \to & \bigvee_{c^X_i} \mathbb{S}^{i+1} \\ \downarrow \widetilde{f}_{i+1} & & \downarrow f_{i+1}/f_i & & \downarrow \Sigma f_i & & \downarrow \Sigma f_i/f_{i-1} & & \downarrow \Sigma \widetilde{f_i} \\ \bigvee_{c^Y_{i+1}} \mathbb{S}^{i+1} & \to & Y_{i+1}/Y_i & \to & \Sigma Y_i & \to & \Sigma(Y_i/Y_{i-1}) & \to & \bigvee_{c^Y_i} \mathbb{S}^{i+1} \end{array}$$

This is immediate: the leftmost and rightmost squares commute by construction of our functorial action, and the middle squares commute by definition.

Thus, we have shown that any cellular map $f_* : X_* \to Y_*$ gives rise to a chain map between the cellular chain complexes of X_* and Y_*. Due to space constraints, we omit the proofs that this operation satisfies the two functor axioms, but we note that they are very direct. This results in a functor $\mathsf{cellChain} : \mathsf{CWstr} \to \mathsf{Ch}$. If we compose this functor with the nth homology functor $H_n : \mathsf{Ch} \to \mathsf{AbGrp}$, we obtain a functorial definition of reduced cellular homology for CW structures. We denote the resulting functor by $\widetilde{H}_n^{\mathsf{str}} : \mathsf{CWstr} \to \mathsf{AbGrp}$.

C. The homology of a CW complex

Our end goal is to extend our cellular homology functor to the category of CW complexes. To do so, we follow the strategy laid out in Figure 1: first, we will need a lemma to show that cellular homology is homotopy invariant.

Proposition 31. *Let f_* and g_* be two parallel cellular maps. Every cellular homotopy between f_* and g_*, induces a chain homotopy between $\mathsf{cellChain}(f_*)$ and $\mathsf{cellChain}(g_*)$.*

The proof is standard but somewhat technical. Due to space constraints, we omit it and refer to the computer formalisation. Proposition 31 implies that $\mathsf{cellChain}$ descends to a functor from $\mathsf{Ho}(\mathsf{CWstr})$ to $\mathsf{Ho}(\mathsf{CW})$. As we already saw, the chain homology functor H_n factors through $\mathsf{Ho}(\mathsf{CWstr})$, meaning that we can compose it with $\mathsf{cellChain}$ to express cellular homology as a functor $\widetilde{H}_n^{\mathsf{str}} : \mathsf{Ho}(\mathsf{CWstr}) \to \mathsf{AbGrp}$. Therefore, we have established that cellular homology is homotopy invariant.

In fact, its definition makes it clear that $\widetilde{H}_n^{\mathsf{str}}(X_*)$ only depends on the $(n+1)$-skeleton of X_*, so $\widetilde{H}_n^{\mathsf{str}}$ can actually be defined as a functor from $\mathsf{Ho}(\mathsf{CWstr}^{(n+1)})$ to AbGrp. Since abelian groups form a univalent category, $\widetilde{H}_n^{\mathsf{str}}$ can even be extended to the Rezk completion of $\mathsf{Ho}(\mathsf{CWstr}^{(n+1)})$, which is $\mathsf{CW}^{(n+1)}$. Composing the resulting functor with the truncation functor from CW to $\mathsf{CW}^{(n+1)}$ yields the desired definition of the cellular homology functor $\widetilde{H}_n^{\mathsf{cw}} : \mathsf{CW} \to \mathsf{AbGrp}$.

D. The Eilenberg-Steenrod axioms

To be deserving of the title of a *homology theory*, our definition should satisfy the Eilenberg-Steenrod axioms. However, this raises yet another constructivity issue: the classical formulation of these axioms involves wedge sums indexed by arbitrary sets, which do not exist in our category of CW structures. To remedy this, we will define a *finitary* version of the axiom[1]. In what follows, $\mathsf{CWstr}_\star$ denotes the category of pointed CW structures.

Definition 32 (Eilenberg-Steenrod homology)**.** *A **reduced homology theory** is a $\mathbb{Z}$-indexed family of functors $\widetilde{E}_n : \mathsf{CWstr}_\star \to \mathsf{AbGrp}$ satisfying the following axioms.*

Suspension: *For any n, there is an isomorphism $\widetilde{E}_n(X_*) \cong \widetilde{E}_{n+1}((\Sigma X)_*)$ which is natural in X.*

Exactness: *For any cellular map f_*, the sequence*

$$\widetilde{E}_n(X_*) \xrightarrow{\widetilde{E}_n(f_*)} \widetilde{E}_n(Y_*) \xrightarrow{\widetilde{E}_n(\mathsf{cfcod}_*)} \widetilde{E}_n((C_f)_*)$$

is exact, meaning that $\ker \widetilde{E}_n(\mathsf{cfcod}_*) = \operatorname{Im} \widetilde{E}_n(f_*)$.

Dimension: *$\widetilde{E}_n(\mathbb{S}^0_*)$ is trivial for $n \neq 0$ and isomorphic to $\mathbb{Z}$ when $n = 0$.*

Binary additivity: *For any $X_*, Y_* : \mathsf{CWstr}_\star$, the canonical map $\widetilde{E}_n(X_*) \oplus \widetilde{E}_n(Y_*) \to \widetilde{E}_n((X \vee Y)_*)$ is an isomorphism.*

An important point is that we decided to define the Eilenberg-Steenrod axioms over $\mathsf{CWstr}_\star$ rather than $\mathsf{CW}_\star$. The reason for this is that the exactness axiom involves cofibres of arbitrary maps, which are not guaranteed to exist in the category $\mathsf{CW}_\star$ (see the discussion around Corollary 15). Nevertheless, we do get a restricted exactness axiom for $\widetilde{H}_n^{\mathsf{cw}}$ which involves only maps with a finite domain as a consequence of the exactness of $\widetilde{H}_n^{\mathsf{str}}$. Before we prove exactness, however, let us show that the suspension axiom is satisfied.

Proposition 33. *The suspension axiom is satisfied by $\widetilde{H}_n^{\mathsf{str}}$.*

Proof. Let (C^X_*, ∂^X_*) and $(C^\Sigma_*, \partial^\Sigma_*)$ be the augmented chain complexes associated to X_* and $(\Sigma X)_*$ respectively. Let $(\widehat{C}^\Sigma_*, \widehat{\partial}^\Sigma_*) := (C^\Sigma_{*+1}, \partial^\Sigma_{*+1})$ be the latter complex shifted by 1, and denote its chain homology groups by $\widetilde{H}^\Sigma_n$. We have $\widetilde{H}^\Sigma_n = \widetilde{H}^{\mathsf{str}}_{n+1}((\Sigma X)_*)$ by construction. We construct a chain map $\varphi_* : \widehat{C}^\Sigma_* \to C^X_*$ as follows:

$$\begin{array}{ccccccccccc}
 & & \mathbb{Z}[c_n] & & & & \mathbb{Z}[c_0] & & \mathbb{Z}[2] & & \\
 & & \| & & & & \| & & \| & & \\
\cdots & \xrightarrow{\widehat{\partial}_{n+1}} & \widehat{C}^\Sigma_n & \xrightarrow{\widehat{\partial}_n} & \cdots & \xrightarrow{\widehat{\partial}_1} & \widehat{C}^\Sigma_0 & \xrightarrow{\widehat{\partial}_0} & \widehat{C}^\Sigma_{-1} & \xrightarrow{\widehat{\partial}_{-1}} & \mathbb{Z} \\
 & & \downarrow{\scriptstyle \varphi_n} & & & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_{-1}} & & \downarrow{\scriptstyle \varphi_{-2}} \\
\cdots & \xrightarrow[\partial_{n+1}]{} & C^X_n & \xrightarrow[\partial_n]{} & \cdots & \xrightarrow[\partial_1]{} & C^X_0 & \xrightarrow[\partial_0]{} & C^X_{-1} & \longrightarrow & \mathbb{1} \\
 & & \| & & & & \| & & \| & & \\
 & & \mathbb{Z}[c_n] & & & & \mathbb{Z}[c_0] & & \mathbb{Z} & &
\end{array}$$

We simply set φ_n to be the identity when $n \geq 0$, and let φ_{-1} be the map forgetting the second generator. The fact that the squares commute is a direct consequence of Proposition 29, apart from the second square from the right whose commutativity follows by construction of $\widehat{\partial}_0$. Thus φ_n induces an isomorphism $\phi_n : \widetilde{H}^{\mathsf{str}}_{n+1}((\Sigma X)_*) = \widetilde{H}^\Sigma_n \to \widetilde{H}^{\mathsf{str}}_n(X_*)$ on homology when $n \geq 1$. Naturality is immediate as the isomorphism is induced by the identity. When $n = 0$, we need to be somewhat more careful since $\widehat{\partial}_0$ and ∂_0 have different codomains. Nonetheless, their kernels trivially agree and so we still obtain the desired isomorphism on homology. The final non-trivial case we need to check is when $n = -1$. This case amounts to showing that $\widetilde{H}^\Sigma_{-1}$ is trivial which follows immediately by construction of $\widehat{\partial}_0$ and $\widehat{\partial}_{-1}$. □

Let us continue with the exactness axiom. For this, we need to characterise the behaviour of the boundary map on pushouts. The proof, which we have to omit here due to space constraints, proceeds by unfolding the definition of the attaching maps in Definition 9 and some direct but tedious computations.

[1] It is also possible to allow for more general families of sets, as done by e.g Cavallo [16] and Buchholtz and Favonia [5]. We choose the finitary version for the sake of simplicity but note that the proofs differ little, should one wish to be more general.

Lemma 34. *Let P_* be the cellular pushout of some span $Y_* \xleftarrow{f_*} X_* \xrightarrow{g_*} Z_*$. The boundary map ∂^P_{n+1} factors as*

$$C^p_{n+1} \xrightarrow{\sim} C^X_n \oplus C^Y_{n+1} \oplus C^Z_{n+1} \xrightarrow{\partial'_{n+1}} C^X_{n-1} \oplus C^Y_n \oplus C^Z_n \xrightarrow{\sim} C^P_n$$

where $\partial'_{n+1}(x, y, z) := (-\partial^X_n\, x, \partial^Y_{n+1}\, y + \overline{f}_n\, x, \partial^Z_{n+1}\, z - \overline{g}_n\, x)$.

Proposition 35. *The exactness axiom is satisfied by $\widetilde{H}^{\mathsf{str}}_n$.*

Proof. The fact that $\mathrm{Im}\,(\widetilde{H}^{\mathsf{str}}_n(f_*)) \subseteq \ker\,(\widetilde{H}^{\mathsf{str}}_n(\mathsf{cfcod}_*))$ follows from the functoriality of $\widetilde{H}^{\mathsf{str}}_n$ and the fact that $\mathsf{cfcod}_* \circ f_*$ is constant by definition of $(C_f)_*$. For the other direction, let $[y] : \widetilde{H}^{\mathsf{str}}_n(Y_*)$ be an equivalence class (where $y : C^Y_n$) and assume it is in the kernel of the composite map $C^Y_n \xhookrightarrow{\overline{\mathsf{cfcod}}_n} C^{\mathsf{cof}}_n \xrightarrow{q} C^{\mathsf{cof}}_n / \partial^{\mathsf{cof}}_{n+1}$. A quick computation reveals that the group C^{cof}_n is equal to $C^X_{n-1} \oplus C^Y_n \oplus \mathbb{1}_n$, and that $\overline{\mathsf{cfcod}}_n\, y$ is equal to $(0, y, 0)$. Therefore, our assumption is equivalent to $(0, y, 0)$ being in the image of $\partial^{\mathsf{cof}}_{n+1}$. Using Lemma 34, this means that $y = \partial^Y_{n+1}\, y_0 + \overline{f}_n\, x$ for some $y_0 : C^Y_{n+1}$ and $x : C^X_n$. Thus, $[y] = [\overline{f}_n\, x]$ in $\widetilde{H}^{\mathsf{str}}_n(Y_*)$. Since $[\overline{f}_n\, x] = \widetilde{H}^{\mathsf{str}}_n(f_*)[x]$, we are done. □

Proposition 36. *The dimension axiom is satisfied by $\widetilde{H}^{\mathsf{str}}_n$.*

Proof. The augmented chain complex associated to $\mathbb{S}^0$ is

$$\ldots \xrightarrow{\partial_3} \mathbb{1} \xrightarrow{\partial_2} \mathbb{1} \xrightarrow{\partial_1} \mathbb{Z}[2] \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\partial_{-1}} \mathbb{1} \xrightarrow{\partial_{-2}} \mathbb{1} \xrightarrow{\partial_{-3}} \ldots$$

The homology of this complex is clearly concentrated in degree 0 with $\widetilde{H}^{\mathsf{str}}_n(\mathbb{S}^0) \cong \mathbb{Z}$. □

Proposition 37. *Binary additivity is satisfied by $\widetilde{H}^{\mathsf{str}}_n$.*

Proof. The direct sum $\widetilde{H}^{\mathsf{str}}_n(X_*) \oplus \widetilde{H}^{\mathsf{str}}_n(Y_*)$ can be viewed as the homology of the chain complex $(C^X_n \oplus C^Y_n, \partial^X_n \oplus \partial^Y_n)$. Under this identification, the map $\widetilde{H}^{\mathsf{str}}_n(X_*) \oplus \widetilde{H}^{\mathsf{str}}_n(Y_*) \to \widetilde{H}^{\mathsf{str}}_n((X \vee Y)_*)$ corresponds to the chain map

$$\begin{array}{ccccccc}
\ldots \longrightarrow & C^Y_n \oplus C^Z_n & \xrightarrow{\partial^Y_n \oplus \partial^Z_n} & C^Y_n \oplus C^Z_n & \longrightarrow \ldots \\
& \downarrow & \quad 0 \oplus \partial^Y_n \oplus \partial^Z_n & \downarrow & \\
\ldots \longrightarrow & C^{\mathbb{1}}_n \oplus C^Y_{n+1} \oplus C^Z_{n+1} & \longrightarrow & C^{\mathbb{1}}_{n\text{-}1} \oplus C^Y_n \oplus C^Z_n & \longrightarrow \ldots
\end{array}$$

where the bottom row is the reduced cell complex associated to $(X \vee Y)_*$ using the cell structure for pushouts. The computation of the boundary map comes from Lemma 34. As $C^{\mathbb{1}}_n$ vanishes for $n \geq 2$, the vertical maps are isomorphisms for in these dimensions and hence we obtained the desired isomorphism of homology in groups. The additional 0-cell in $(X \vee B)_0$ affects the computation forces us to construct the inverse of the prospective isomorphism explicitly in dimensions $n = 1$ and $n = 0$. The construction is completely standard and we refer to the computer formalisation for details. □

Theorem 38. *The functor $\widetilde{H}^{\mathsf{str}}_n : \mathsf{CWstr}_\star \to \mathsf{AbGrp}$ is an ordinary reduced homology theory.*

Finally, we arrive at the corresponding result for $\widetilde{H}^{\mathsf{cw}}_n$ (where the notion of exactness is restricted to mention only those pushouts which exist in $\mathsf{CW}_\star$).

Corollary 39. *The functor $\widetilde{H}^{\mathsf{cw}}_n : \mathsf{CW}_\star \to \mathsf{AbGrp}$ is an ordinary reduced homology theory.*

Some care has to be taken when inferring Corollary 39 from Theorem 38. As $\widetilde{H}^{\mathsf{cw}}_n$ concerns the homology of arbitrary types *merely* equipped with a CW structure, we are only able to automatically infer the two axioms which happen to be propositions, namely exactness and binary additivity. The dimension axiom follows because it concerns $\mathbb{S}^0$, a closed type for which we have an explicit CW structure. Finally, we need to take care of the suspension axiom. Its statement is a set and not a proposition, which prevents us from using the usual elimination principle for truncations, but we can instead use the set elimination principle of Kraus [17, Chapter 8.1.1]. We need to prove the theorem whenever X has an explicit CW structure (using Theorem 38), and then show that the proof does not depend on the choice of CW structure, which is a direct consequence of naturality.

V. Part 4: The Hurewicz Theorem

As previously mentioned, homology groups are quite similar in spirit to homotopy groups, so one might hope that the two notions are connected in some way. The answer lies in the Hurewicz theorem, which states that if a space is n-connected, then its homology groups coincide with its homotopy groups up to dimension $n+1$ (up to abelianisation in the case $n = 0$).

A. Approximating n-connected spaces

The classical proof of the Hurewicz theorem for cellular homology takes an arbitrary n-connected CW complex, and replaces its CW structure with an alternative one with no nontrivial cells in dimension $< n+1$. This is done by defining the new set of $(n+1)$ cells to be the $(n+1)$-th homotopy group of the space, from which the Hurewicz theorem will follow. Unfortunately, this approach will not work in our framework since the homotopy groups of a finite CW complex may very well be infinite (think of the n-dimensional sphere). Yet, perhaps surprisingly, we are able to give a constructive proof of the Hurewicz theorem by using a different construction for the alternative structure of n-connected CW complexes.

Definition 40. *We say that a CW structure X_* is* ***Hurewicz n-connected*** *if $c^X_0 = 1$ and $c^X_i = 0$ for $0 < i < n$. We use the same terminology for CW complexes which merely have a Hurewicz n-connected CW structure.*

We remark that being Hurewicz n-connected is a property (i.e. a proposition). The following lemma gives a few elementary consequences of Hurewicz n-connectedness.

Lemma 41. *Let X_* be a CW structure. If X_* is Hurewicz n-connected, then*

1) *$X_i \simeq \mathbb{1}$ for $0 \leq i < n$*
2) *$X_{n+1} \simeq \bigvee_{\mathsf{Fin}(c_{n+1})} \mathbb{S}^{n+1}$.*
3) *X_i is n-connected for $i \in \{j \in \mathbb{N} \cup \{\infty\} \mid j > n\}$*

Item 3 tells us that Hurewicz n-connectedness implies the usual notion of n-connectedness. The other direction is much less obvious – especially constructively. Nonetheless, we can, in fact, prove it. As a warm up, let us tackle the case $n = 0$.

Proposition 42. *For any* 0*-connected structure* X_**, there is a Hurewicz* 0*-connected CW structure* X'_* *s.t.* $X_i = X'_i$ *for* $i \geq 1$*.*

Proof. We proceed by induction on c_0^X, i.e. the size of X_0 ($\simeq \mathsf{Fin}(c_0^X)$). If $c_0^X = 0$, this contradicts the 0-connectedness of X_*. If $c_0^X = 1$, X_* is already of the right form and there is nothing to prove. Consider now the case $c_0^X > 1$. We will be done if we can show that X_1 may be obtained as the pushout of $\mathsf{Fin}(c'_0) \xleftarrow{\alpha'} \mathbb{S}^0 \times \mathsf{Fin}(c'_1) \xrightarrow{\mathsf{snd}} \mathsf{Fin}(c'_1)$ for some $c'_0, c'_1 : \mathbb{N}$ and α' satisfying $c'_0 < c_0^X$. Let us carry out the construction. Some of the arguments may look non-constructive but we emphasise that they are justified as they concern finite sets.

First, note that there must be some $a_0 : \mathsf{Fin}(c_1)$ such that $\alpha_0(\mathsf{north}, a_0) \neq \alpha_0(\mathsf{south}, a_0)$. Indeed, if this were not the case, we would have that $\|X_1\|_0 \simeq X_0$. By combining this equation with $\|X_\infty\|_0 \simeq \|X_1\|_0$, we would obtain that $\|X_\infty\|_0$ is isomorphic to X_0, which is not contractible since $c_0^X > 1$. Now, by permuting the elements of $\mathsf{Fin}(c_1)$ and $\mathsf{Fin}(c_0)$, we may assume that the last element $a_0 : \mathsf{Fin}(c_1)$ satisfies $\alpha_0(\mathsf{north}, a_0) = c_0^X - 1$ and $\alpha_0(\mathsf{south}, a_0) = c_0^X - 2$. We define a new attaching map $\alpha'_0 : \mathbb{S}^0 \times \mathsf{Fin}(c_1^X - 1) \to \mathsf{Fin}(c_0^X - 1)$ by

$$\alpha'_0(x, y) = \begin{cases} \alpha_0(x, y) \text{ if } \alpha_0(x, y) < c_0 - 1 \\ c_0^X - 2 \text{ otherwise} \end{cases}$$

The 1-skeleton X'_1 obtained by pushing out along α'_0 is easily identified with X_1, and thus we are done as we have decreased the cardinality of the codomain of the attaching map by 1. ☐

Before turning to higher dimensions, let us define a useful alteration of the notion of CW structure. In what follows, we abuse notation for the sake of convenience and interpret $\bigvee_A \mathbb{S}^{-1}$ as the empty type rather than than the unit type.

Definition 43. *A **good** CW structure is a pointed CW structure* X_* *whose attaching maps* $\alpha_i : \mathsf{Fin}(c_{i+1}^X) \times \mathbb{S}^i \to X_i$ *lift to maps defined over sphere bouquets, i.e. for all* i *there exists* $\alpha'_i : \bigvee_{\mathsf{Fin}(c_{i+1}^X)} \mathbb{S}^i \to_\star X_i$ *such that* $X_{i+1} \simeq C_{\alpha_i^X}$*.*

Lemma 44. *Let* X_* *be a good CW structure.* X_* *is Hurewicz* n*-connected iff* $X_{n+1} \simeq \bigvee_B \mathbb{S}^{n+1}$ *and* $X_{n+2} \simeq C_f$ *where* A *and* B *are finite types and* $f : \bigvee_A \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}$*.*

This lemma follows immediately from the definition of good structures and Lemma 41. We remark that good CW structures always are 0 Hurewicz connected. The converse also holds for simple connectedness reasons.

Proposition 45. *Any finite Hurewicz* 0*-connected CW structure is merely good.*

We are now ready to prove the main technical theorem of which states that the synthetic standard notion of connectedness coincides, for CW complexes, with the more analytic notion of Hurewicz connectedness.

Theorem 46. *Let* X_* *be a an* n*-connected CW structure. There merely exists a Hurewicz* n*-connected CW structure* X'_* *such that* $X_i = X'_i$ *for* $i > n$*.*

Proof. We proceed by induction on n. The base case is given by Proposition 42. For the inductive step, let X_* be n-connected. In particular, X_* is $(n-1)$-connected, so by induction hypothesis we may assume that it is Hurewicz $(n-1)$-connected. Since $n > 0$, this structure is also Hurewicz 0-connected and we may assume that it is good (up to some fixed finite dimension $k >> n$) by Proposition 45. Using Lemma 44, we know that $X_n \simeq \bigvee_A \mathbb{S}^n$ and $X_{n+1} \simeq C_f$ for $f : \bigvee_B \mathbb{S}^n \to \bigvee_A \mathbb{S}^n$ where A and B are some finite sets. Using Lemma 44 again, we are done if we can find construct sets A', B' and $f' : \bigvee_{B'} \mathbb{S}^{n+1} \to \bigvee_{A'} \mathbb{S}^{n+1}$ s.t. $X_{n+2} \simeq C_{f'}$. Consider the following diagram where $C = \mathsf{Fin}(c_{n+2}^X)$.

$$\begin{array}{ccccc}
 & & \bigvee_C \mathbb{S}^{n+1} & \longrightarrow & \mathbb{1} \\
 & & \downarrow{\scriptstyle \alpha_{n+1}} & \ulcorner & \downarrow \\
\bigvee_A \mathbb{S}^n & \xrightarrow{\mathsf{cfcod}} & C_f & \longrightarrow & X_{n+2} \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
\mathbb{1} & \longrightarrow & \bigvee_B \mathbb{S}^{n+1} & \longrightarrow & X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}
\end{array}$$

The top square is a pushout square because $X_*^{(k)}$ is a good CW structure (we have identified X_{n+1} with C_f). The fact that the bottom-left square is a pushout follows by the first part of Lemma 2. The bottom right-square is less evident. Consider the composite map $\bigvee_A \mathbb{S}^n \to X_{n+2}$ on the second row. Using the fact that X_∞ (and hence also X_{n+2}) is n-connected, it is an easy consequence of Lemma 4 that this map is merely constant. As we are proving a proposition, we may ignore the word 'merely' and assume that it is constant. This means that the composition of the two bottom squares is a pushout by the second part of Lemma 2. Consequently, the the bottom-right square is also a pushout square. Let us write β for the map $\bigvee_C \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}$ that is described by the middle column of the diagram. We have shown that $C_\beta \simeq X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$. Another way to interpret this equivalence is that we gave the space $X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$ a good Hurewicz n-connected CW structure V, with $(n+1)$-skeleton $V_{n+1} = \bigvee_B \mathbb{S}^{n+1}$ and attaching map $\alpha_{n+1} = \beta$.

Consider the inclusion $\mathsf{inr} : \bigvee_A \mathbb{S}^{n+1} \to X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$. This happens to be a map between CW complexes, so we may approximate it using Theorem 14. Doing so produces a map $\mathsf{inr}_{n+1} : \bigvee_A \mathbb{S}^{n+1} \to V_{n+1} = \bigvee_B \mathbb{S}^{n+1}$, which factors inr as $\bigvee_A \mathbb{S}^{n+1} \xrightarrow{\mathsf{inr}_{n+1}} \bigvee_B \mathbb{S}^{n+1} \xrightarrow{\iota_{n+1}} X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$. Now consider the following diagram.

$$\begin{array}{ccccc}
\bigvee_A \mathbb{S}^{n+1} & \xrightarrow{\mathsf{inr}_{n+1}} & \bigvee_B \mathbb{S}^{n+1} & \longrightarrow & X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1} \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
\mathbb{1} & \longrightarrow & C_{\mathsf{inr}_{n+1}} & \longrightarrow & X_{n+2}
\end{array}$$

The left square is a pushout by definition, and the total square is a pushout for elementary reasons. Thus, the right square is a pushout. Replacing $X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$ with C_β, we conclude that X_{n+2} is obtained as the pushout of the span $C_{\mathsf{inr}_{n+1}} \leftarrow \bigvee_B \mathbb{S}^{n+1} \rightarrow C_\beta$. An application of the 3×3 lemma tells us that this is equivalent to cofibre of the map $\mathsf{inr}_{n+1} \vee \beta : \bigvee_{A+C} \mathbb{S}^{n+1} \rightarrow \bigvee_B \mathbb{S}^{n+1}$. Thus, we have shown that X_{n+2} is of the desired form and we are done. □

Corollary 47 (The Hurewicz Approximation Theorem)**.** *A CW complex is n-connected iff it is Hurewicz n-connected.*

B. From homotopy to homology

In order to state our final theorem, we will need the help of the Hurewicz homomorphism. We define it using $\widetilde{H}^{\mathsf{cw}}_n$, but remark that the construction carries over to $\widetilde{H}^{\mathsf{str}}_n$.

Definition 48. *Let X be a CW complex. Define the* ***Hurewicz homomorphism***[2] $\eta : \pi_n(X) \rightarrow \widetilde{H}^{\mathsf{cw}}_n(X)$ *on canonical elements $f : \mathbb{S}^n \rightarrow_\star X$ by letting $\eta(|f|) : \widetilde{H}^{\mathsf{cw}}_n(X)$ be the image of 1 under the composition $\mathbb{Z} \xrightarrow{\sim} \widetilde{H}^{\mathsf{cw}}_n(\mathbb{S}^n) \xrightarrow{f_*} \widetilde{H}^{\mathsf{cw}}_n(X)$.*

The Hurewicz theorem will provide us with a condition for when this homomorphism is an isomorphism. Before we state and prove it, let us try to understand the groups involved in the 'simple' special case when X is the cofibre C_f of some map of (finite) sphere bouquets $f : \bigvee_A \mathbb{S}^n \rightarrow \bigvee_B \mathbb{S}^n$. This special case will turn out to inform the proof for the general case. As C_f has an explicit CW structure, let us switch our homology theory to $\widetilde{H}^{\mathsf{str}}_n$. Now let us compute $\widetilde{H}^{\mathsf{str}}_n(C_f)$ using the exactness axiom: consider the sequence

$$\bigvee_A \mathbb{S}^n \xrightarrow{f} \bigvee_B \mathbb{S}^n \xrightarrow{\mathsf{cfcod}} C_f \xrightarrow{\mathsf{cfcod}} C_{(\mathsf{cfcod}:\bigvee_B \mathbb{S}^n \rightarrow C_f)} \simeq \bigvee_A \mathbb{S}^{n+1}$$

where the final equivalence is the usual characterisation of X_{n+1}/X_n using that C_f has a CW structure. This is a cofibre sequence, and so the following sequence is exact

$$\widetilde{H}^{\mathsf{str}}_n\left(\bigvee_A \mathbb{S}^n\right) \xrightarrow{f_*} \widetilde{H}^{\mathsf{str}}_n\left(\bigvee_B \mathbb{S}^n\right) \xrightarrow{\mathsf{cfcod}_*} \widetilde{H}^{\mathsf{str}}_n(C_f) \rightarrow 0 \qquad (1)$$

where the final 0 comes from that fact that $\widetilde{H}^{\mathsf{str}}_n$ vanishes on $\bigvee_A \mathbb{S}^{n+1}$. We can compute the first two homology groups using additivity, and thus we see that $\widetilde{H}^{\mathsf{str}}_n(C_f) \cong \mathbb{Z}[B]/\mathbb{Z}[A]$. Let us now compute the domain of η, i.e. the group $\pi_n(C_f)$.

Proposition 49. *For any $f : \bigvee_A \mathbb{S}^n \rightarrow \bigvee_B \mathbb{S}^n$ where $n \geq 1$ and A and B are finite types, there is an exact sequence*

$$\pi_n\left(\bigvee_A \mathbb{S}^n\right) \xrightarrow{f_*} \pi_n\left(\bigvee_B \mathbb{S}^n\right) \xrightarrow{\mathsf{cfcod}_*} \pi_n(C_f).$$

Proof. This follows from the Seifert-Van Kampen theorem [1, Example 8.7.17] in the case $n = 1$, and from the Blakers-Massey theorem [18] in the case $n > 1$. For details, we refer to the formalisation. □

We are now almost ready for the Hurewicz theorem. In order to state it, let us define π^{ab}_n to be the abelianisation of the homotopy group functor, i.e. $\pi^{\mathsf{ab}}_n(X) := \pi_n(X)/\operatorname{Im}[-,-]$ where $[-,-] : \pi_n(X) \times \pi_n(X) \rightarrow \pi_n(X)$ is the commutator defined by $[x, y] = xyx^{-1}y^{-1}$. As higher homotopy groups are already abelian, the quotient map $\pi_n(X) \rightarrow \pi^{\mathsf{ab}}_n(X)$ is an isomorphism; in what follows, we will simply interpret π^{ab}_n as π_n when $n \geq 2$. We will, with some abuse of notation, view the Hurewicz homomorphism η as being defined over π^{ab}_n. This is justified as the codomain is an abelian group.

[2]The fact that this map is a homomorphism boils down to the easy fact that the multiplication of cellular maps $\mathbb{S}^n \rightarrow_\star X$ is again cellular.

Theorem 50. *The Hurewicz homomorphism $\eta : \pi^{\mathsf{ab}}_n(X) \rightarrow \widetilde{H}^{\mathsf{cw}}_n(X)$ is an isomorphism for any $(n-1)$-connected CW complex X.*

Proof. Since we are proving a proposition, we can assume that we have a CW structure X_* and switch our homology theory to $\widetilde{H}^{\mathsf{str}}_*$. Since the map $X_{n+1} \rightarrow X_\infty$ is n-connected, the canonical map $\pi_n(X_{n+1}) \rightarrow \pi_n(X)$ is an equivalence. Similarly, $\widetilde{H}^{\mathsf{str}}_n(X_\bullet) = \widetilde{H}^{\mathsf{str}}_n(X^{(n+1)}_\bullet)$ by definition. Thus, it suffices to show the theorem for the $(n+1)$-skeleton of X. As X is Hurewicz $(n-1)$ connected we may assume that $X_n = \bigvee_B \mathbb{S}^n$ and that $X_{n+1} = C_f$ for some $\alpha : \bigvee_A \mathbb{S}^n \rightarrow \bigvee_B \mathbb{S}^n$. Elementary algebra tells us that abelianisation is right-exact and thus preserves the exact sequence in Proposition 49. Let us compare this sequence (top sequence below) to the corresponding for homology in (1) (bottom sequence below).

$$\begin{array}{ccccc}
\pi^{\mathsf{ab}}_n\left(\bigvee_A \mathbb{S}^n\right) & \xrightarrow{f_*} & \pi^{\mathsf{ab}}_n\left(\bigvee_B \mathbb{S}^n\right) & \xrightarrow{\mathsf{cfcod}_*} & \pi^{\mathsf{ab}}_n(C_f) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathbb{Z}[A] & \xrightarrow{\overline{f}} & \mathbb{Z}[B] & \longrightarrow & \mathbb{Z}[B]/\mathbb{Z}[A] \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\widetilde{H}^{\mathsf{str}}_n\left(\bigvee_A \mathbb{S}^n{}_\bullet\right) & \xrightarrow{f_*} & \widetilde{H}^{\mathsf{str}}_n\left(\bigvee_B \mathbb{S}^n{}_\bullet\right) & \xrightarrow{\mathsf{cfcod}_*} & \widetilde{H}^{\mathsf{str}}_n((C_f)_\bullet)
\end{array}$$

The isomorphisms $\pi^{\mathsf{ab}}_n(\bigvee_C \mathbb{S}^n) \cong \mathbb{Z}[C]$ for $C \in \{A, B\}$ come from the inclusion $\bigvee_A \mathbb{S}^n \rightarrow \Pi_A(\mathbb{S}^n)$ which induces an isomorphism on homotopy groups when $n \geq 2$; this also happens to be an isomorphism on π^{ab}_1. On homology, the isomorphism is a direct consequence of the Eilenberg-Steenrod axioms (but can also be obtained by simply inspecting the related chain complex). The fact that the two left-most squares commute holds almost by definition of the maps involved. Hence we obtain an isomorphism $\pi^{\mathsf{ab}}_n(C_f) \cong \widetilde{H}^{\mathsf{str}}_n(C_f)_\bullet$. We simply have to verify that this isomorphism is equal to η. It is enough to check this on the inclusion of generators from $\pi^{\mathsf{ab}}_n(\bigvee_B \mathbb{S}^n)$ – but here there is nothing to prove: simply unfolding the definitions involved, it is immediate that the desired equality holds. □

VI. Conclusions and Future Work

We hope the reader is now convinced that the theory of CW complexes and cellular homology has a home in HoTT. The fact that the results we have proved in this paper – in particular the cellular and Hurewicz approximation theorems – are at all provable without any form of choice was initially a surprise to us. The theory of CW complexes and cellular homology as it is developed classically often 'feels' constructive, with many constructions being inductive, but it makes heavy use of choice principles. An important takeaway is that this feeling is justified: a significant part of this theory *is* constructive.

However, the initial motivation behind this project was not to carry out a case study in constructive mathematics.

Originally, our development was motivated by the recent proof of the Serre Finiteness theorem by Barton and Campion [19]. This proof relies on homology computations and the Hurewicz theorem, thus the formalisation that accompanies this paper should be helpful to the ongoing formalisation of the Serre Finiteness theorem (by, in particular, Milner [20]).

This paper also aims to be integrated into a larger project including Mörtberg, which seeks to use cellular (co)homology to reduce homological arguments in HoTT to concrete computations which we can run in proof assistants. The canonical example is the computation of the *Brunerie number* [2], a number whose value is given by a certain cohomology computation which, as it is constructively defined in HoTT, should simply be produced by evaluating it in a proof assistant, but whose evaluation is computationally infeasible. Our hope is that if these computations are ported to a cellular (co)homology theory, these many of them should become feasible, paving the way for *proofs by computation* in HoTT.

It would also be interesting to use our cellular approach to explore more advanced results and constructions such as the Steenrod squares and the (currently open) Künneth formula.

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