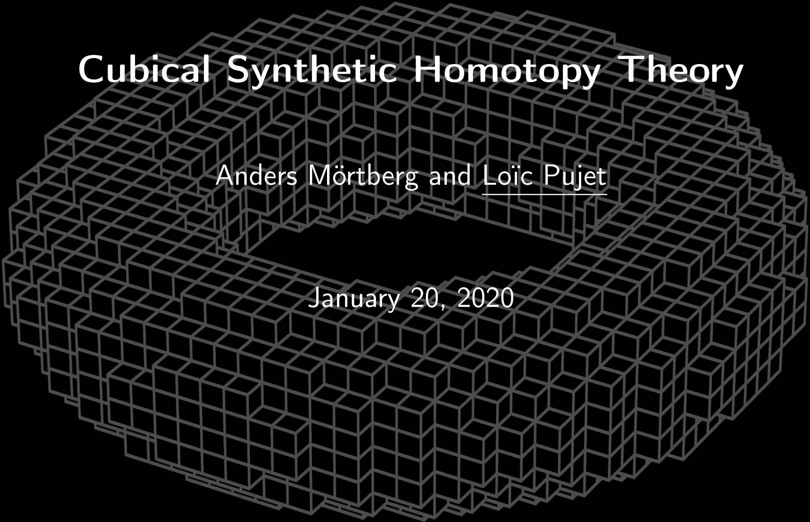

Cubical Synthetic Homotopy Theory

Anders Mörtberg and Loïc Pujet

January 20, 2020

Homotopy Type Theory

Synthetic reasoning about spaces through dependent type theory

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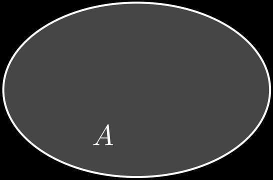
Synthetic reasoning about spaces through dependent type theory

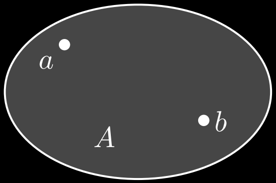
- A : Type



Types are spaces

Synthetic reasoning about spaces through dependent type theory

- A : Type
- a, b : A

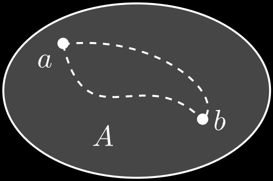


Terms are points

Homotopy Type Theory

Synthetic reasoning about spaces through dependent type theory

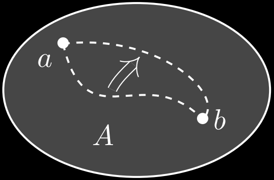
- A : Type
- a, b : A
- e, f : $a \equiv_A b$



(Propositional) equality proofs are paths

Synthetic reasoning about spaces through dependent type theory

- $A : \text{Type}$
- $a, b : A$
- $e, f : a \equiv_A b$
- $h : e \equiv_{a \equiv b} f$



Equalities between equalities are homotopies

Synthetic reasoning about spaces through dependent type theory

- $A : \mathrm{Type}$
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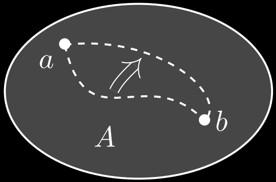


And so on...

Homotopy Type Theory

relies on

relies on

Univalence Axiom

$$(A \equiv B) \simeq (A \simeq B)$$

"Equalities between types are equivalences"

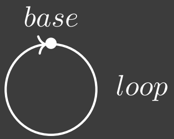
relies on

Univalence Axiom

$$(A \equiv B) \simeq (A \simeq B)$$

"Equalities between types are equivalences"

Higher Inductive Types



$\texttt{data}\ S^1 : \text{Type}\ \texttt{where}$

$\quad base : S^1$

$\quad loop : base \equiv base$

Homotopy Type Theory

relies on

Univalence Axiom

$$(A \equiv B) \simeq (A \simeq B)$$

"Equalities between types are equivalences"

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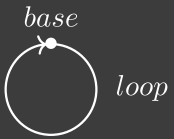

$$\begin{aligned} &\texttt{data}\ S^1 : \text{Type}\ \texttt{where} \\ &\quad \mathit{base} : S^1 \\ &\quad \mathit{loop} : \mathit{base} \equiv \mathit{base} \end{aligned}$$

Which do not have nice computational content in HoTT.

Cubical Type Theory

Take the topological intuitions **seriously**

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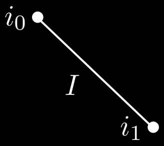
Interval type I with two endpoints $i_0, i_1 : I$



Take the topological intuitions **seriously**

Interval type I with two endpoints
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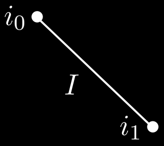
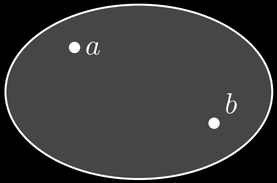
A proof of $a \equiv_A b$ is a function
$e : I \to A$
$e\ i_0 = a \qquad e\ i_1 = b$





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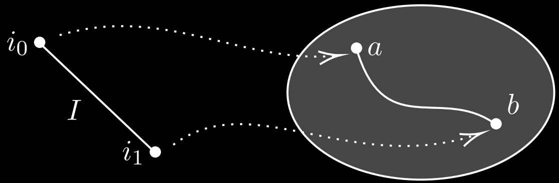

Cubical Type Theory

Replace the elimination principle with **composition** and **transport**

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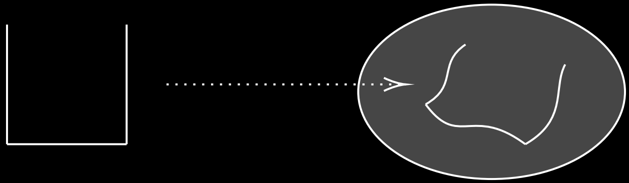
Composition : from three matching paths $I \to A$ forming an open square, build a map $I \times I \to A$



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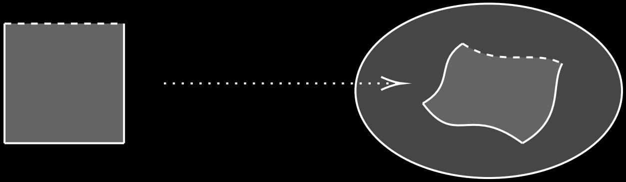

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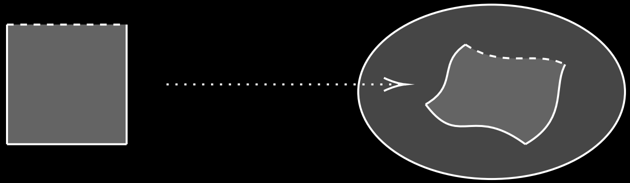
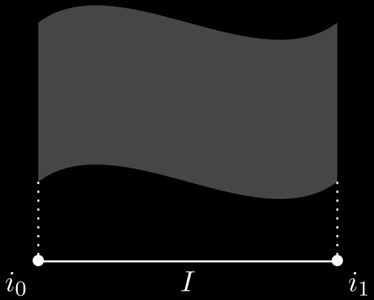


Likewise, from five matching maps $I \times I \to A$ forming an open cube, build a map $I \times I \times I \to A$, and so on.

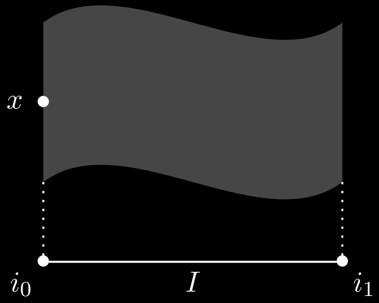
Replace the elimination principle with **composition** and **transport**

Transport : given a type family $P : I \to \mathrm{Type}$



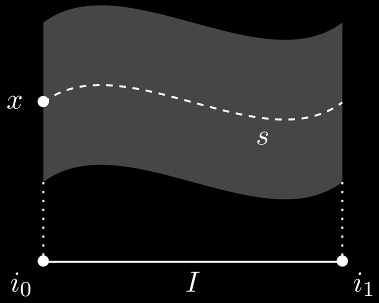
Replace the elimination principle with **composition** and **transport**

Transport : given a type family $P : I \to \mathrm{Type}$ and a point $x : P\ i_0$,



Replace the elimination principle with **composition** and **transport**

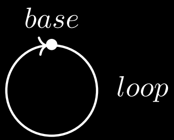
Transport : given a type family $P : I \to \mathrm{Type}$ and a point $x : P\ i_0$, build a term $s : (i : I) \to P\ i$.



Cubical Type Theory

Satisfies **univalence** without breaking computational properties.
(using *Glue* types)

Provides computational **higher inductive types**.



$$loop\ i_0 = base$$
$$loop\ i_1 = base$$

definitionally

```
data S¹ : Type where
  base : S¹
  loop : base ≡ base
```

Higher inductive types have simpler elimination principles.

Suppose we have a family $P : S^1 \to \mathcal{U}$, and we want to build a term of type $(x : S^1) \to P\ x$.

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HoTT

$$\frac{\Gamma \vdash b : P\ \mathit{base} \qquad \Gamma \vdash \ell : \mathrm{transport}\ P\ \mathit{loop}\ b \equiv b}{\Gamma \vdash \_ : (x : S^1) \to P\ x}$$

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Suppose we have a family $P : S^1 \to \mathcal{U}$, and we want to build a term of type $(x : S^1) \to P\ x$.

HoTT

$$\frac{\Gamma \vdash b : P\ \mathit{base} \qquad \Gamma \vdash \ell : \mathrm{transport}\ P\ \mathit{loop}\ b \equiv b}{\Gamma \vdash \_ : (x : S^1) \to P\ x}$$

CuTT

$$\frac{\Gamma \vdash b : P\ \mathit{base} \qquad \Gamma, i \vdash \ell : P\ (\mathit{loop}\ i)}{\Gamma \vdash \_ : (x : S^1) \to P\ x}$$

Synthetic homotopy theory

A good chunk of homotopy theory has been formalized with HoTT

- The torus is the product of two circles
- The sphere is the suspension of the circle
- The 3×3 lemma for pushouts
- The fundamental group of the circle is $\mathbb{Z}$
- The van Kampen theorem
- The Blakers-Massey theorem
- The Freudenthal suspension theorem
- The Hopf Fibration
- Cohomology and Eilenberg-Mac Lane spaces
- etc...

Synthetic homotopy theory

A good chunk of homotopy theory has been formalized with **CuTT**

- **The torus is the product of two circles**
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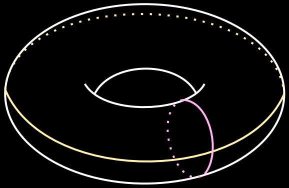
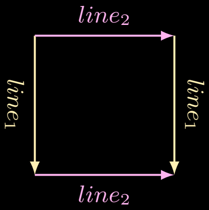
Case study: the torus





$$
\begin{aligned}
&\texttt{data}\ T^2\ :\ \text{Type}\ \texttt{where}\\
&\quad \mathit{point}\ :\ T^2\\
&\quad \mathit{line}_1\ :\ \mathit{point} \equiv \mathit{point}\\
&\quad \mathit{line}_2\ :\ \mathit{point} \equiv \mathit{point}\\
&\quad \mathit{square}\ :\ \text{PathP}\ (\lambda\, i \to \mathit{line}_1\ i \equiv \mathit{line}_1\ i)\ \mathit{line}_2\ \mathit{line}_2
\end{aligned}
$$

A. Vezzosi, A. Mörtberg, A. Abel (2019).
Cubical Agda: A dependently typedProgramming Language with Univalence and Higher Inductive Types

Case study: the torus

We want to prove $T^2 \equiv S^1 \times S^1$.

Case study: the torus

We want to prove $T^2 \simeq S^1 \times S^1$.

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$$f : T^2 \to S^1 \times S^1 \qquad s : f \circ g \equiv id$$
$$g : S^1 \times S^1 \to T^2 \qquad r : g \circ f \equiv id$$

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We want to prove $T^2 \simeq S^1 \times S^1$.

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$$s : f \circ g \equiv id$$
$$r : g \circ f \equiv id$$

$$f \;:\; T^2 \to S^1 \times S^1$$
$$f \;x \;=\; ?$$

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$$
\begin{aligned}
&f \;:\; T^2 \to S^1 \times S^1 \\
&f\ point && = ? \\
&f\ (line_1\ i) && = ? \\
&f\ (line_2\ j) && = ? \\
&f\ (square\ i\ j) && = ?
\end{aligned}
$$

Case study: the torus

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$g : S^1 \times S^1 \to T^2$

$s : f \circ g \equiv id$

$r : g \circ f \equiv id$

$$
\begin{aligned}
& f \;:\; T^2 \to S^1 \times S^1 \\
& f \; point && = \; (base, base) \\
& f \; (line_1 \; i) && = ? \\
& f \; (line_2 \; j) && = ? \\
& f \; (square \; i \; j) && = ?
\end{aligned}
$$

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$$
\begin{aligned}
&f \ : T^2 \to S^1 \times S^1 \\
&f \ point && = \ (base, base) \\
&f \ (line_1 \ i) && = \ (loop \ i, base) \\
&f \ (line_2 \ j) && = ? \\
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$f : T^2 \to S^1 \times S^1$

$g : S^1 \times S^1 \to T^2$

$s : f \circ g \equiv id$

$r : g \circ f \equiv id$

$$g \; : \; S^1 \times S^1 \to T^2$$
$$g \; x = \; ?$$

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$$s : f \circ g \equiv id$$
$$r : g \circ f \equiv id$$

$$g \ : \ S^1 \times S^1 \to T^2$$
$$g \ (x, y) \ = ?$$

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$$\begin{aligned}
g \ &: \ S^1 \times S^1 \to T^2 \\
g \ (base, base) \quad &= ? \\
g \ (loop \ i, base) \quad &= ? \\
g \ (base, loop \ j) \quad &= ? \\
g \ (loop \ i, loop \ j) \quad &= ?
\end{aligned}$$

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$$
\begin{aligned}
& g \;:\; S^1 \times S^1 \to T^2 \\
& g\ (base, base) && = \; point \\
& g\ (loop\ i, base) && = ? \\
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\begin{aligned}
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$g : S^1 \times S^1 \to T^2$

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$$
\begin{aligned}
s \, : \, (x \, : \, S^1 \times S^1) \to \; & f \; (g \; x) \equiv x \\
s \; (base, base) \; & = \; refl \\
s \; (loop \; i, base) \; & = \; refl \\
s \; (base, loop \; j) \; & = \; refl \\
s \; (loop \; i, loop \; j) \; & = \; refl
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$$

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$$\begin{aligned}
r \ : \ (x \ : \ T^2) \to \ & \ g \ (f \ x) \equiv x \\
r \ point \quad & = \ refl \\
r \ (line_1 \ i) \quad & = \ refl \\
r \ (line_2 \ j) \quad & = \ refl \\
r \ (square \ i \ j) \quad & = \ refl
\end{aligned}$$

Case study: the torus

The proofs of this theorem in pure HoTT are about 150 lines.

Some comparison

	Coq	Lean	Agda	Cubical
$\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$	150	-	150	25
$\Omega(\mathbb{S}^1) = \mathbb{Z}$	160	80	90	50
3×3 lemma	-	-	3000	200
Join assoc	-	230	210, 320	240

Conclusion

HoTT

- Inductive equality
- J rule
- Unwieldy HITs
- Univalence as an axiom

CuTT

- Interval type
- Composition and transport
- Pattern-matching for HITs
- Fully computational

- Easier treatment of synthetic homotopy theory, especially with HITs
- New, topology-oriented ways to construct proofs and spaces
- The Cubical-Agda library is catching up on most HoTT milestones
- We can compute the Brunerie number... in theory