Rules of Dependent Type Theory

Universes and contexts

$$\textsc{Ctx-Empty}\quad \frac{}{\bullet\ \text{context}} \qquad \textsc{Ctx-Ext}\quad \frac{\Gamma\ \text{context} \qquad \Gamma \vdash A : \mathsf{Type}_i}{(\Gamma, x : A)\ \text{context}} \qquad \textsc{Assumption}\quad \frac{(t : A) \in \Gamma}{\Gamma \vdash t : A}$$

$$\textsc{Conversion}\quad \frac{\Gamma \vdash t : A \qquad A \equiv A'}{\Gamma \vdash t : A'} \qquad \textsc{Univ-Form}\quad \frac{\Gamma\ \text{context}}{\Gamma \vdash \mathsf{Type}_i : \mathsf{Type}_{i+1}}$$

Dependent products / Π-types

$$\textsc{Pi-Form}\quad \frac{\Gamma \vdash A : \mathsf{Type}_i \qquad \Gamma, x : A \vdash B : \mathsf{Type}_j}{\Gamma \vdash \Pi\,(x : A)\,.\,B : \mathsf{Type}_{\max(i,j)}}$$

$$\textsc{Pi-Intro}\quad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda\,(x : A)\,.\,t : \Pi\,(x : A)\,.\,B} \qquad \textsc{Pi-Elim}\quad \frac{\Gamma \vdash t : \Pi\,(x : A)\,.\,B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B[u/x]}$$

Computation rules:

- $(\lambda\,x\,.\,t)\ u \equiv t[u/x]$
- $t \equiv \lambda\,x\,.\,(t\ x)$ (when t has type $\Pi\,(x : A)\,.\,B$)

NB: when the codomain B does not depend on A, we write $A \to B$ instead of $\Pi\,(x : A)\,.\,B$.

Dependent sums / Σ-types

$$\textsc{Sigma-Form}\quad \frac{\Gamma \vdash A : \mathsf{Type}_i \qquad \Gamma, x : A \vdash B : \mathsf{Type}_j}{\Gamma \vdash \Sigma\,(x : A)\,.\,B : \mathsf{Type}_{\max(i,j)}}$$

$$\textsc{Sigma-Intro}\quad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/x]}{\Gamma \vdash \langle\, t\,, u\,\rangle : \Sigma\,(x : A)\,.\,B} \qquad \textsc{Sigma-Elim1}\quad \frac{\Gamma \vdash t : \Sigma\,(x : A)\,.\,B}{\Gamma \vdash \pi_1(t) : A} \qquad \textsc{Sigma-Elim2}\quad \frac{\Gamma \vdash t : \Sigma\,(x : A)\,.\,B}{\Gamma \vdash \pi_2(t) : B[\pi_1(t)/x]}$$

Computation rules:

- $\pi_1(\langle\, t\,, u\,\rangle) \equiv t$
- $\pi_2(\langle\, t\,, u\,\rangle) \equiv u$
- $t \equiv \langle\, \pi_1(t)\,, \pi_2(t)\,\rangle$ (when t has type $\Sigma\,(x : A)\,.\,B$)

NB: when the second type B does not depend on A, we write $A \times B$ instead of $\Sigma\,(x : A)\,.\,B$.

Natural numbers

$$\frac{\Gamma \text{ context}}{\Gamma \vdash \mathbb{N} : \mathsf{Type}_0}\ \textsc{Nat-Form} \qquad \frac{\Gamma \text{ context}}{\Gamma \vdash 0 : \mathbb{N}}\ \textsc{Nat-Intro1} \qquad \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{suc}(t) : \mathbb{N}}\ \textsc{Nat-Intro2}$$

$$\frac{\Gamma \vdash P : \mathbb{N} \to \mathsf{Type}_i \qquad \Gamma \vdash p_0 : P\ 0 \qquad \Gamma \vdash p_s : \Pi\,(n : \mathbb{N})\,.\,P\ n \to P\ (\mathsf{suc}\ n) \qquad \Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathbb{N}\mathsf{ind}\,(P, p_0, p_s, n) : P\ n}\ \textsc{Nat-Elim}$$

Computation rules:

- $\mathbb{N}\mathsf{ind}\,(A, p_0, p_s, 0) \quad \equiv \quad p_0$
- $\mathbb{N}\mathsf{ind}\,(A, p_0, p_s, \mathsf{suc}(n)) \quad \equiv \quad p_s\ n\ (\mathbb{N}\mathsf{ind}\,(A, p_0, p_s, n))$

Martin-Löf's identity type

$$\frac{\Gamma \vdash A : \mathsf{Type}_i \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{Id}\,(A, t, u) : \mathsf{Type}_i}\ \textsc{Id-Form} \qquad \frac{\Gamma \vdash A : \mathsf{Type}_i \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathsf{refl}\,(t) : \mathsf{Id}\,(A, t, t)}\ \textsc{Id-Intro}$$

$$\frac{\begin{array}{c}\Gamma \vdash A : \mathsf{Type}_i \qquad \Gamma \vdash t : A \\ \Gamma \vdash P : \Pi\,(x : A)\,.\,\mathsf{Id}\,(A, t, x) \to \mathsf{Type}_i \qquad \Gamma \vdash p : P\ t\ (\mathsf{refl}\,(t)) \qquad \Gamma \vdash u : A \qquad \Gamma \vdash e : \mathsf{Id}\,(A, t, u)\end{array}}{\Gamma \vdash \mathsf{J}\,(A, t, P, p, u, e) : P\ u\ e}\ \textsc{Id-Elim}$$

Computation rule: $\mathsf{J}\,(A, t, P, p, t, \mathsf{refl}\,(t)) \quad \equiv \quad p$

Sum types

$$\frac{\Gamma \vdash A : \mathsf{Type}_i \qquad \Gamma \vdash B : \mathsf{Type}_j}{\Gamma \vdash A + B : \mathsf{Type}_{\max(i,j)}}\ \textsc{Sum-Form} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{in}_1(t) : A + B}\ \textsc{Sum-Intro1} \qquad \frac{\Gamma \vdash t : B}{\Gamma \vdash \mathsf{in}_2(t) : A + B}\ \textsc{Sum-Intro2}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : (A + B) \to \mathsf{Type}_i \\ \Gamma, x : A \vdash p_1 : P\ (\mathsf{in}_1(x)) \qquad \Gamma, x : B \vdash p_2 : P\ (\mathsf{in}_2(x)) \qquad \Gamma \vdash t : A + B\end{array}}{\Gamma \vdash \begin{array}{l}\mathsf{match}\ t\ \mathsf{with} \\ \mid \mathsf{in}_1(x) \mapsto p_1 \\ \mid \mathsf{in}_2(x) \mapsto p_2\end{array} : P\ t}\ \textsc{Sum-Elim}$$

Computation rules:

- $\mathsf{match}\ \mathsf{in}_1(t)\ \mathsf{with} \mid \mathsf{in}_1(x) \mapsto p_1 \mid \mathsf{in}_2(x) \mapsto p_2 \quad \equiv \quad p_1[t/x]$
- $\mathsf{match}\ \mathsf{in}_2(t)\ \mathsf{with} \mid \mathsf{in}_1(x) \mapsto p_1 \mid \mathsf{in}_2(x) \mapsto p_2 \quad \equiv \quad p_2[t/x]$

Empty type

$$\frac{\Gamma \vdash t : \bot}{\Gamma \vdash \mathrm{raise}(t) : A}\ \textsc{Empty-Elim}$$

No computation rules!

Unit type

$$\frac{}{\Gamma \vdash \star : \top}\ \textsc{Unit-Intro}$$

Computation rule: $\quad t \equiv \star \qquad$ (when t has type $\top$)