OTT

Observational Equality

meets

The Calculus of Inductive Constructions

CIC

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10 april 2024

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Both Coq and Lean are based on the CIC

- Dependent type theory
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- an impredicative sort for propositions
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But difficulties with function extensionality and quotient types

Observational Equality

In observational type theories[1] the inductive equality is replaced with the observational equality:

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \sim_A u : SProp}$$

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$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \sim_A u : SProp}$$

Observational equality is eliminated via typecasting:

$$\frac{\Gamma \vdash e : A \sim B \qquad \Gamma \vdash t : A}{\Gamma \vdash cast(A, B, e, t) : B}$$

which computes by case analysis on A and B.

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- introduction rules
- elimination rules
- computation rules

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In observational type theory, every type former is also equipped with

- a definition for the equality between inhabitants
- a definition for the equality between two instances of the type
- computation rules for type-casting

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- A computation rule for type-casting

$$\begin{aligned} & cast(A \to B, C \to D, e, f) \\ & \quad \equiv \quad \lambda\, (x : C) \,.\, cast(B, D, e_2, f\, cast(C, A, e_1, x)) \end{aligned}$$

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How do we fit the general inductive definitions of CIC into this picture?

First Example: Lists

$$
\begin{array}{l}
\mathit{Inductive}\ \ \mathit{list}\ \ (A : \mathit{Type}_\ell) : \mathit{Type}_\ell := \\
\mid \mathit{nil} : \mathit{list}\, A \\
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$$\begin{aligned}&\mathit{cast}(\mathit{list}\,A, \mathit{list}\,B, e, \mathit{nil}) &&\equiv\ \mathit{nil} \\ &\mathit{cast}(\mathit{list}\,A, \mathit{list}\,B, e, \mathit{cons}\ a\ l) &&\equiv \\ &\qquad \mathit{cons}\ \ \mathit{cast}(A, B, \mathit{list\text{-}eq}\ e, a)\ \ \mathit{cast}(\mathit{list}\,A, \mathit{list}\,B, e, l)\end{aligned}$$

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  Does not typecheck ☹

Observational Inductives?

Not so simple!

Constructor arguments, not parameters!

The universe inconsistency shows up because the size of an inductive is determined by the types of its constructor arguments, not parameters or indices.

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$$\mid \mathit{small} : \mathbb{N} \to \mathit{Small}\ A$$

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With this technique, we can smoothly handle all inductive definitions without indices

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Treating indices will require a few more tricks.

No Canonicity for Indices

Remember our failed attempt at a computation rule

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We cannot simplify casts on indices in general...

...but we can encode them away with observational equality

"You can pick any colour, as long as it is black"

Henry Ford's trick[2] can be used to encode indices with equalities on parameters:

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and now we can used our recipe for inductives without indices.

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In the case of the inductive equality, Henry Ford's encoding produces:

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Our strategy: present eq to the user but elaborate everything to eq_F under the hood.

$$\begin{array}{rcl} eq & \rightsquigarrow & eq_F \\ eq\_refl & \rightsquigarrow & eq\_refl_F\ refl \\ & \ldots & \end{array}$$

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$$eq\_elim \quad \rightsquigarrow \quad \ldots$$

We can write a term with the expected type using $cast$ and eq_F_elim

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The missing ingredient is the computation rule for cast on reflexivity:

$$cast(A, A, refl, t) \equiv t$$

The missing rule

Our goal: adding $cast(A, A, refl, t) \equiv t$ as a definitional equality

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Because of proof irrelevance, it should apply whenever the two endpoints of the cast are convertible:

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$\rightarrow$ nonlinear reduction rule which specifies reduction mutually with conversion checking

Is this déjà vu?

This idea is reminiscent of Lean's treatment of the J eliminator:

$$\frac{P\ a \equiv P\ b}{J(A, a, P, t, b, e) \equiv t}$$

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This is not an undecidability result, though

> Does the addition of Werner's rule, while destroying proof normalization, preserve decidability of conversion and type checking? (Since proofs are irrelevant for equality, they need not be normalized during type checking.)

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input: t and u
(neutral terms)

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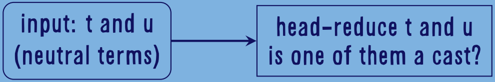

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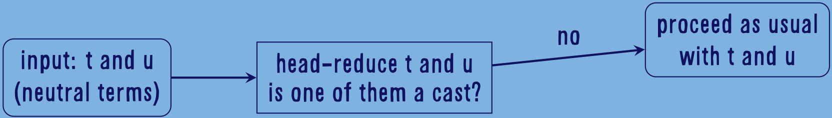

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Because cast reduces on type constructors, this rule only plays a role for relevant neutral terms.

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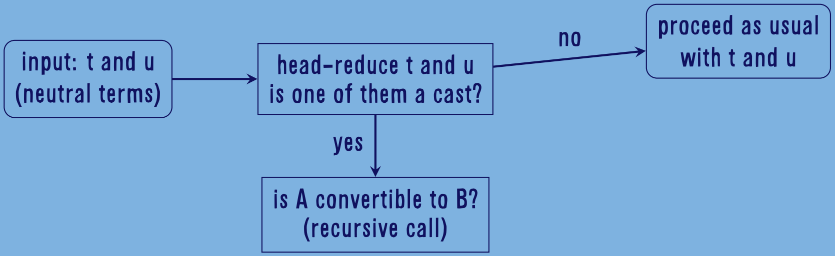

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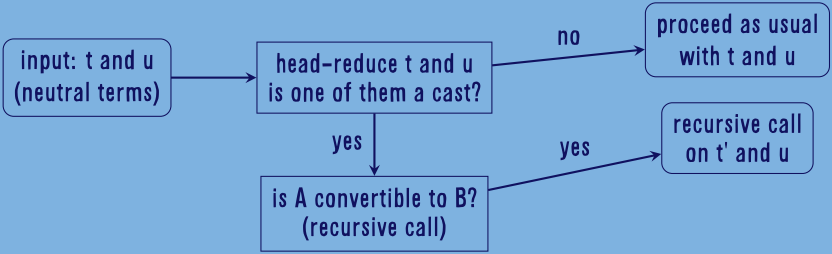

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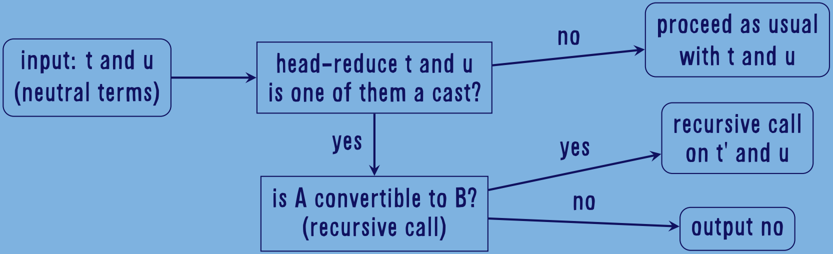

Decidability proof

Using the second approach, we can add it on top of our logical relation model for $\mathsf{TT}^{obs}/\mathsf{CC}^{obs}$ [3]

[3] P, Tabareau '22 – Obserational Equality: Now for good

Decidability proof

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→ Formal Agda proof of the decidability of conversion for our new rule

[3] P, Tabareau '22 – Obserational Equality: Now for good

Coming soon-ish

In your favourite rooster-themed proof assistant!

```
Set Observational Inductives.

(* Declaring an inductive automaticall adds equalities and rewrite rules for cast *)
Inductive list (A : Type) : Type :=
| nil : list A
| cons : forall (a : A) (l : list A), list A.

Parameter A B : Type.
Parameter e : list A ~ list B.
Parameter a : A.

Eval cbn in (cast (list A) (list B) e [ a ]).
(* [ cast A B (obseq_cons_0 A B e) a ] *)
```