Cubical Coq Using Intensional Presheaves

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Abstract. Cubical type theory provides a computational interpretation of the univalence principle, based on cubical presheaf semantics. Presheaf models are usually formulated inside an extensional meta-theory, but using an alternative definition of presheaves that behaves well in intensional type theory, Pédrot formulated presheaf models as a translation from type theory to itself. We make use of this translation to enrich intensional type theory with the Orton and Pitts axioms for cubical type theory, giving them a computational meaning. As everything is done constructively, this preserves the good computational properties of intensional type theory, such as decidability of type checking. In particular, the translation of integers derived using the axioms compute to actual integers. This is a first step towards making cubical type theory compute inside Coq, resulting in a new modular framework for consistency and canonicity proofs of cubical type theories.

Keywords: Homotopy Type Theory · Cubical Type Theory · Constructive Mathematics · Coq.

1 Introduction

The univalence principle was suggested by Voevodsky in [12] as a new axiom for dependent type theory, implying various interesting extensionality features. Along with higher inductive types, it is one of the corner stones of Homotopy Type Theory (HoTT), an extension of Martin-Löf type theory.

In order to give semantics for HoTT, Voevodsky initially put forward the so-called simplicial model [6], which successfully models univalence in simplicial sets but crucially relies on several non-constructive principles [4]. The search for constructive semantics later resulted in the development of Cubical Type Theory [3], a fully constructive and computational dependent type theory that satisfies the univalence principle.

Cubical Type Theory has been implemented in various proof assistants (namely Cubical Agda [10], cubicaltt, RedPRL, redtt etc), and has been successfully used to give computational proofs of various results using the univalence principle [7].

In this paper, we attempt to reproduce the cubical semantics from [3], but formalizing everything in intuitionistic type theory as a program translation. We argue that doing so provides us with two main benefits. On the one hand, it gives a fully verified proof of consistency for HoTT assuming only the consistency of a variant of dependent type theory. On the other hand, it results in a new way

to compute with univalence, along with a fully verified proof of canonicity: every closed term of type $\mathbb{N}$ reduces to a canonical integer. Moreover, such proofs are quite modular, and should readily be adapted to extensions of cubical type theory.
At the time of writing, this is still a work in progress. We explain the ongoing state of the formalization, the challenges encountered, and how we hope to handle the remaining challenges.

2 Cubical Presheaf models

Cubical type theory was initially designed as an internal language for the cubical model [2]. This model is an instance of a presheaf model, which is a generic way to model dependent type theory in a category of functors $\mathrm{Set}^{\mathcal{C}^{op}}$ (see [5] for a detailed account). The main feature of this model is that it satisfies univalence in a constructive meta-theory.

In [9], Pédrot gave a way to formulate presheaf models as a translation in intuitionistic type theory. More precisely, given a strict category $\mathbb{P}$ (by strict, we mean that the associativity and unitality axioms hold definitionally), Pédrot defines a syntactic translation from CIC (the Calculus of Inductive Constructions) to a theory called sCIC, which is similar to CIC but where the impredicative universe of propositions is replaced by a predicative hierarchy of universes of *strict* propositions (denoted `SProp`) satisfying a definitional version of Uniqueness of Identity Proof. One of the main features of this translation is its modularity: assuming canonicity of sCIC (which is not yet proven, although it is expected to hold), and some very mild assumption on $\mathbb{P}$, the type theory generated by the translation still satisfies canonicity. In particular, if one can extend CIC with additional axioms and constructively extend Pédrot's translation to these axioms, then one gets a canonicity result for this extended CIC.

The goal of this project is, for a suitable choice of category $\mathbb{P}$, to extend Pédrot's translation to CIC+Univalence. Thanks to the aforementioned canonicity result, this would provide a new way to compute with univalence.

Our choice of $\mathbb{P}$ is dictated by the existing cubical presheaf models of univalence. More precisely, we follow [3], and take for $\mathbb{P}$ the category of cubes described there. The objects of our category $\mathbb{P}$ are therefore integers, and the class of arrows $p \leq q$ is a subset of the maps in $(\underline{p} \to \mathbb{B}) \to (\underline{q} \to \mathbb{B})$, where $\underline{p}$ denotes the p-element set (encoded as the set of natural numbers strictly smaller than p):

```
Record cube_arr (p q : nat) : Set := {
  arr : (finset p → 𝔹 ) → (finset q → 𝔹 ) ;
  eps_arr : increasing arr }.
```

The `increasing` predicate, defined in `SProp`, asserts that for any $f : \underline{p} \to \mathbb{B}$ and $x : \underline{p}$, if $f\,x = \mathrm{true}$ then $\mathrm{arr}\,f\,x = \mathrm{true}$. Composition of two such maps is induced by the usual composition of maps, which is strictly associative and unital: since `eps_arr` is irrelevant, this makes $\mathbb{P}$ into a strict category, as required to apply Pédrot's translation.

3 Orton and Pitts Axioms

In the initial paper that makes use of cubical presheaves , Cohen *et al* describe a model of univalence [3]. Later, Orton and Pitts [8] abstracted away the main argument of [3] and gave a list of nine axioms which are sufficient for a topos to satisfy univalence.

Since, in an extensional theory, cubical presheaves form a topos satisfying those axioms, we decided as a proof of concept to first prove Orton and Pitts axioms in our model, before moving on to univalence. Note that Orton and Pitt's proof takes place in an extensional type theory, and so we cannot get in this way an intensional proof of univalence. Nevertheless, since Orton and Pitts formalized their development in Agda, proving the nine axioms would provide the first fully formal consistency proof for univalence (without universes). Both cubical type theory and Orton and Pitts axioms require the presence of a path type $\mathbb{I}$, which behaves as an interval. In our cubical presheaf model, $\mathbb{I}$ is defined as the Yoneda functor (which is a map $\mathbb{P} \to \forall p : \mathbb{P}, Type_p$) evaluated at 1. Extensionally, a term $t : \mathbb{I}$ therefore consists of an object t_p of type $p \leq 1$ for any $p : \mathbb{P}$, plus some naturality conditions. In particular, $\mathbb{I}$ contains two terms $\mathbf{0}$ and $\mathbf{1}$, mapping any $f : \underline{p} \to \mathbb{B}$ to the constant map equal to false (resp. true).

The first axiom of Orton and Pitts states that $\mathbb{I}$ is connected:

$$\forall \phi : \mathbb{I} \to \texttt{Type}, (\forall (i : \mathbb{I}), \phi i \vee \neg \phi i) \to (\forall i : \mathbb{I}, \phi i) \vee (\forall i : \mathbb{I}, \neg \phi i)$$

The proof goes as follows. Suppose first that $\phi\ \mathbf{0}$ holds, and fix $i \in \mathbb{I}_p$ for some $p : \mathbb{P}$. It is possible to construct a map $p + 1 \leq p$, which induces a map $h_p : \mathbb{I}_p \to \mathbb{I}_{p+1}$, such that on its top face, $h_p\ i$ is constant equal to $\mathbf{0}_p$, while on its bottom face, $h_p\ i$ is i. We now distinguish between two cases. If $\phi\ (h_p\ i)$ holds, then so does $\phi\ i$ by naturality of ϕ, which completes the proof. Otherwise, once again by naturality of ϕ, we know that $\phi\ \mathbf{0}$ does not hold, which leads to a contradiction. The case where $\phi\ \mathbf{0}$ does not hold is similar.

The second axiom states that $\mathbb{I}$ is not degenerate: $\neg(\mathbf{0} = \mathbf{1})$. It is easily proved by injectivity of the constructors of $\mathbb{B}$.

Axioms 3 and 4 impose conditions on maps $\sqcup, \sqcap : \mathbb{I} \to \mathbb{I} \to \mathbb{I}$ forming connections on I. Namely, $\sqcap$ behaves like a minimum, and $\sqcup$ as a maximum:

$$\forall i : \mathbb{I}, \mathbf{0} \sqcap i = \mathbf{0} = i \sqcap \mathbf{0} \wedge \mathbf{1} \sqcap i = i = i \sqcap \mathbf{1}$$

$$\forall i : \mathbb{I}, \mathbf{0} \sqcup i = i = i \sqcup \mathbf{0} \wedge \mathbf{1} \sqcup i = \mathbf{1} = i \sqcup \mathbf{1}$$

These axioms boil down to maps $\mathrm{min_p}, \mathrm{max_p} : p \leq 1 \to p \leq 1 \to p \leq 1$ which have to obey the equations analog to axioms 3 and 4. It seems that such maps do not exist intensionally, so we rely instead on axioms asserting that the disjunction || and conjunction && on booleans "compute on both sides". More specifically, we assume that $(\lambda x.\, x \,||\, 0) = (\lambda x.\, x)$ and $(\lambda x.\, x \,||\, 1) = (\lambda x.\, 1)$ (in addition to the computation rules $(\lambda x.\, 0 \,||\, x) \equiv (\lambda x.\, x)$ and $(\lambda x.\, 1 \,||\, x) \equiv (\lambda x.\, 1)$), and similarly for &&. Those equations are for example a simple consequence of functional extensionality. Defining $\min_p$ and $\max_p$ in terms of those parallel || and && then yields a proof of the two axioms.

Orton and Pitts go on to define 5 more axioms. Four of them describe *cofibrant propositions*, that play the same role as the face lattice in cubical type theory. They assume a property $\texttt{cof} : \Omega \to \Omega$ defined on the universe of propositions, that satisfies

$$\begin{array}{ll} \forall i : \mathbb{I}, & \texttt{cof}(i = \mathbf{0}) \wedge \texttt{cof}(i = \mathbf{1}) \\ \forall \phi\ \psi : \Omega, & \texttt{cof}\phi \to \texttt{cof}\psi \to \texttt{cof}(\phi \vee \psi) \\ \forall \phi\ \psi : \Omega, & \texttt{cof}\phi \to (\phi \to \texttt{cof}\psi) \to \texttt{cof}(\phi \wedge \psi) \\ \multicolumn{2}{l}{\forall \phi : \mathbb{I} \to \Omega, (\forall i : \mathbb{I}, \texttt{cof}(\phi i)) \to \texttt{cof}(\forall i : \mathbb{I}, \phi i)} \end{array}$$

In our model, we rendered this as a type F whose translation is the *face lattice presheaf*, which naturally comes equipped with operations corresponding to the ones described in these four axioms.

The last axiom is slightly more complicated. It is analogue to the "gluing" construction of cubical type theory, as it takes a cofibrant proposition ϕ, a partial type $A : \phi \to \texttt{Type}$ and a type B that is equivalent to A when ϕ is true, and extends A in a full type equivalent to B. We haven't succeeded in formally implementing this axiom yet, but we expect that it is doable.

4 Towards computational 2-level type theory

In the cubical and simplicial models, all HoTT terms are interpreted as *fibrant* cubical sets – that is, cubical sets equipped with an additional structure designed to eliminate univalent equality. However, there have been various proposals for a *2-level* type theory [11] [1] that reflects fibrant types as well as non-fibrant ones. It comes with two different equality types: a "strict" equality akin to the usual type-theoretical equality, and a "univalent" equality as in HoTT, which is restricted to fibrant types.

Since our presheaf model contains non-fibrant cubical sets too, we could imagine enriching CIC with axioms mimicking the ones from 2-level type theory, if we can give them a constructive interpretation in our model. Doing so could result in computation rules for 2-level type theory along with a canonicity result, something that does not exist yet to our knowledge.

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