Computing with Univalence in Coq using Intensional Presheaves

Loïc Pujet

July 11, 2020

Homotopy Type Theory

Synthetic reasoning about spaces through dependent type theory

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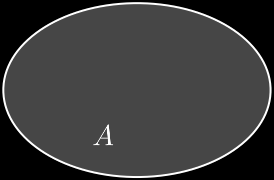
- $A : \mathcal{U}$



Types are spaces

Synthetic reasoning about spaces through dependent type theory

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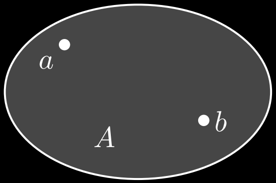

Terms are points

Homotopy Type Theory

Synthetic reasoning about spaces through dependent type theory

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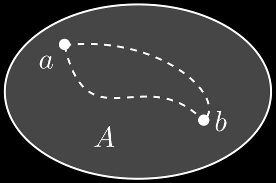

(Propositional) equality proofs are paths

Synthetic reasoning about spaces through dependent type theory

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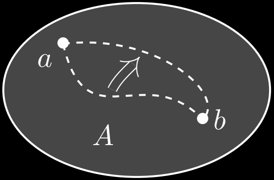

Equalities between equalities are homotopies

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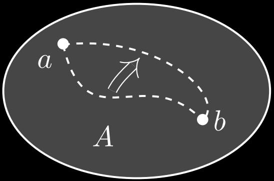

And so on...

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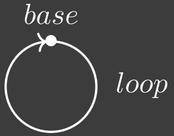

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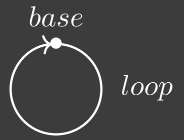

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Which do not have computational content as is.

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Basically what you get by gluing sets of points, lines, squares, cubes, ... along their faces.

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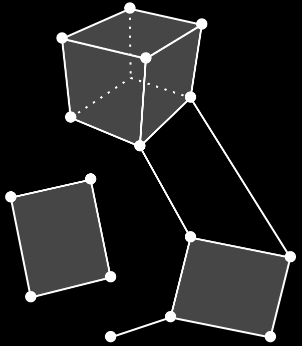

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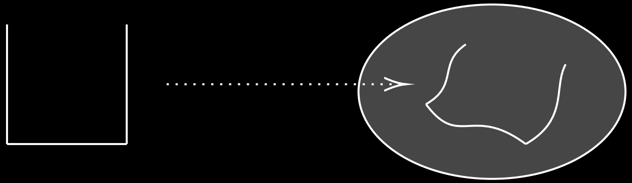

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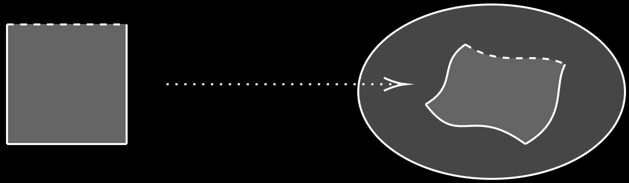

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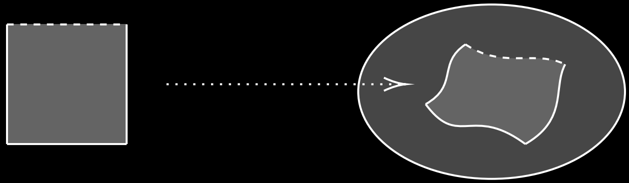

The box filling operation implements the **J eliminator** for the equalities (higher dimensional faces) of the cubical structure.

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It has been reified into **Cubical Type Theory**, which satisfies canonicity and univalence. However, normalization and decidability of type-checking are still open.

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\end{array}$$

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Note that $(\mathcal{S}, \cong)$ does not have a directed reduction, though.

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(unless we are dealing with an extensional target, but then computation breaks)

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Our goal is to do the same with *fibrant* cubical presheaves, to get a translation from HoTT to MLTT+SProp+UIP.

Definitional Presheaves

What has been done, what remains to be done

Dependent Products	✓
Dependent Sums	✓
Booleans	✓
Cubical equality	✓
Function extensionality	✓
J eliminator	✓
Weak univalence	✓
Full univalence	**?**
Fibrancy of the universe	**?**
Higher Inductive Types	**?**

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This method really shines when it comes to extensions of HoTT:

- Good properties for 2-level type theory with little extra work
- You want computational simplicial types? Just define them in the model using strict equality, prove their fibrancy, and voilà!
- etc

Thank you!