Revisiting Hofmann's setoid model

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I. Background

Axioms in type theory

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$$\frac{}{\mathbb{N} : \mathit{Type}} \qquad \frac{}{0 : \mathbb{N}} \qquad \frac{n : \mathbb{N}}{S\,n : \mathbb{N}}$$

$$\frac{P : \mathbb{N} \to \mathit{Type} \qquad p_z : P\,0 \qquad p_s : \Pi\,(n : \mathbb{N}).\,P\,n \to P\,(S\,n)}{\mathit{natrec}(P, p_z, p_s) : \Pi\,(n : \mathbb{N}).\,P\,n}$$

The Hurry Coward correspondance

These derivation rules interact via the conversion relation

See also: cut elimination, program evaluation

$$
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→ conversion should be decidable

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Propositional equality

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Now it becomes possible to state and prove e.g.

- $\Pi\,(n : \mathbb{N})\,(m : \mathbb{N}).\, n + m =_{\mathbb{N}} m + n$
- $\Pi\,(n : \mathbb{N}).\,(n = 0) + (\Sigma\,(m : \mathbb{N}).\, n =_{\mathbb{N}} S\ m)$

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This propositional equality is **under-determined** at higher types

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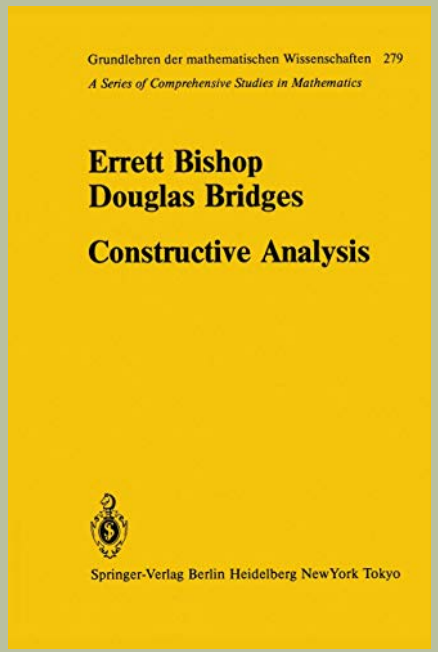
MLTT does not support **quotient types** either

It is not a bug, it is a feature

Per Martin-Löf intended his type theory as an intensional and predicative foundation for Bishop-style constructive mathematics



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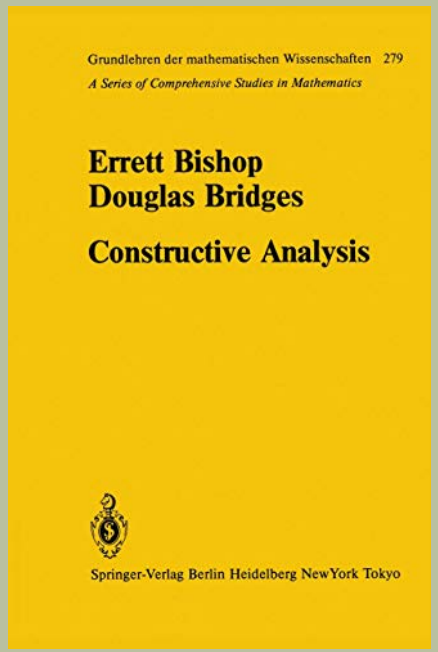

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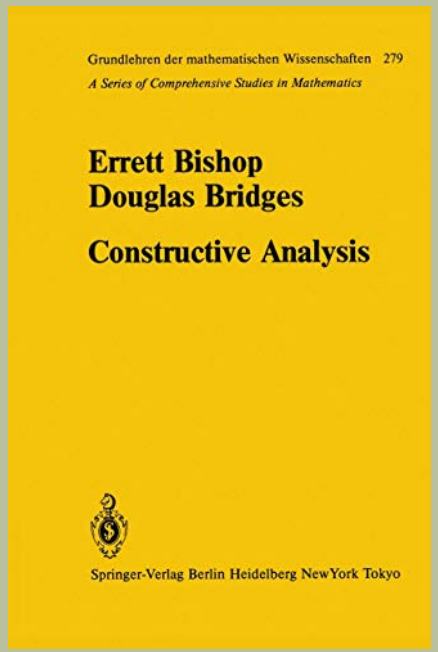

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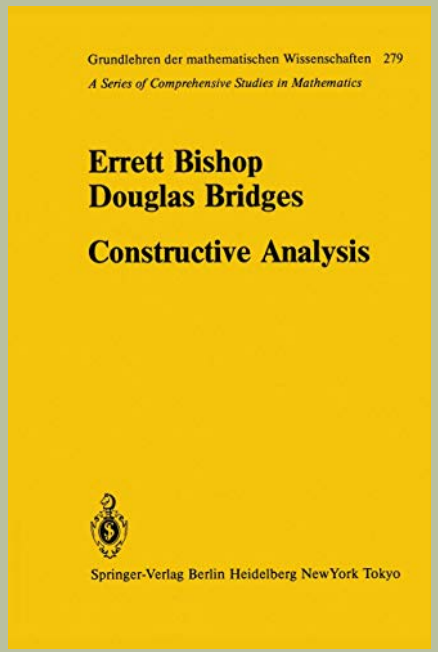

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A setoid morphism is a function which preserves the equivalence relation

Setoid Hell

In practice though, setoids are a bit clunky:

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...so much for axiom-free mathematics!

II. Setoid models

Compilation 101

In his PhD thesis, Martin Hofmann envisioned a compiler for these extensionality axioms

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Contexts $\rightarrow$ Setoids
Substitutions $\rightarrow$ Setoid morphisms
Types $\rightarrow$ Dependent setoids
Terms $\rightarrow$ Sections of dependent setoids

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$[\![funext]\!] \quad \equiv \quad$ (* some proof-term in MLTT *)

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Hofmann's model only handles a weak version of type theory

The source of the problem

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Setoid should really be understood as a *Setoid*-enriched category!
Unfortunately, the rules of type theory are strict

A non-solution

Category theorists will define the ex/reg completion by quotienting the function spaces $A \to B$ by setoid equality

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Constructively, this construction behaves very poorly

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Extend MLTT with a universe of strict propositions

$$\frac{A : SProp \qquad x : A \qquad y : A}{x \equiv y}$$

Then define a strict setoid as a type with a *SProp*-valued relation

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Extend MLTT with a universe of strict propositions

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Then define a strict setoid as a type with a *SProp*-valued relation

We can now interpret types as (dependent) strict setoids

MLTT + extensionality − universes $\longrightarrow$ MLTT + SProp

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Altenkirch, Boulier, Kaposi, Sattler, Sestini 2021

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To do so, they need their propositional equality to live in SProp
Let's call this requirement strict equality

$$\underbrace{\text{MLTT + extensionality}}_{\text{What we wanted}} \longrightarrow \underbrace{\text{MLTT + SProp + strict eq}}_{\text{"Reasonable" extension}}$$

Everything comes at a cost

Are we happy now?

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Let A be a type, and $R : A \to A \to Type$ an equivalence relation
Forming the quotient A/R destroys the computational content of R

$\Rightarrow$ from $\pi(x) =_{A/R} \pi(y)$, we can never recover $R\ x\ y$

III. Revisiting Hofmann's model

Back to the basics

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Define a category *ReflSetoid* whose morphisms preserve reflexivity

Extensionally, *ReflSetoid* does provide a model of MLTT, but

- some equations are propositional instead of definitional
- effectivity of quotients is non-constructive

More categorical jargon

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Setoid = reflexive fibrant presheaves on the truncated semi-cube category

ReflSetoid = fibrant presheaves on the truncated cube category

Perverse sheaves, but even worse

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Pédrot (2020) found an alternative presentation of presheaf models which features strict naturality equations

Putting everything together

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Extra step: eliminate induction-recursion from the construction

What do we gain?

We got back to the result of Altenkirch et al.

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Additionally, we can recover some computational content from equalities using e.g. the accessibility predicate

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Effectivity of quotients amounts to the following principle:

- Let A be a setoid whose elements are all equal (hProp)
- Let $\|A\|$ be the setoid whose equality relation is replaced by the trivial one
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...but it does not preserve reflexivity

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$$\Pi\,(x : A)(y : A).(x =_A y) + (x =_A y \to \bot)$$

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- Reflexivity is preserved!

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Claim
We can give a presentation of MLTT + SProp + strict eq + ELEM which is strongly normalising and has canonicity

Computing with ELEM

Let's enrich type theory with an operator which decides equality

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$\Rightarrow$ This operator is strongly intensional. In presence of function extensionality, equality is not decidable at higher types!

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$eq\ A\ x\ y \rightsquigarrow left$ if x and y are equal closed terms
$eq\ A\ x\ y \rightsquigarrow right$ if x and y are distinct closed terms
$eq\ A\ x\ y$ is neutral otherwise

Final Result

MLTT + ext. + eff. quotients $\longrightarrow$ MLTT + SProp + strict eq + ELEM

Question:
Can we turn this model into a reasonable type theory?

Thank you!