Strictifying Categories with Families

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Today's menu

Formalising normalisation proofs for dependent type theory

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~ with a side of gluing ~

I.

Introduction

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It incorporates computation within the logical foundations

- Mathematical objects are considered up to $\beta\eta$-equality
- Mathematical constructions are programs!

→ The problem of type-checking is now intrinsically linked with computation.

Why bother proving normalisation?

Normalisation property:

Every well-typed terms reduce to a normal form, and the $\beta\eta$-equality of terms corresponds to the syntactical equality of normal forms.

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which are inductively generated by typing rules:

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \Pi (x : A).B} \qquad \frac{\Gamma \vdash t \equiv u : A}{\Gamma \vdash u \equiv t : A} \qquad \ldots$$

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- For complex properties, a naive induction will not go through: we must strengthen the induction hypothesis
- The standard tool for this is logical relations

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- A reducible term of type U...

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- We must account for free variables
  → Reducibility PERs should contain the PER of neutral terms
- We must account for weakening
  → Reducibility PERs should really be presheaves of PERs

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- Barras, Werner, "Coq in Coq" (1997)
- Barras, "Intuitionistic Set Theory and Type Theories with Inductive Families" (2012)
- Wieczorek, Biernacki, "A Coq Formalization of Normalization by Evaluation for Martin-Löf Type Theory" (2018)
- Abel, Öhman, Vezzosi, "Decidability of conversion for type theory in type theory" (2018)
- Adjedj, Lennon-Bertrand, Maillard, Pédrot, P., "Martin-Löf à la coq" (2023)

II.

Gluing? Quésaco?

The algebraist's approach

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A category with families is the data of:

- A category of contexts and subtitutions
- For every context Γ, a set of types $\mathsf{Ty}\ \Gamma$
- For every subst. $\sigma : \Delta \to \Gamma$, a function $\mathsf{Ty}\ \Gamma \to \mathsf{Ty}\ \Delta$
- For every context Γ and type A, a set of terms $\mathsf{Tm}\ \Gamma\ A$
- For every subst. $\sigma : \Delta \to \Gamma$, a function $\mathsf{Tm}\ \Gamma\ A \to \mathsf{Tm}\ \ \Delta\ \ A[\sigma]$
- Context extensions $\Gamma \triangleright A$, context projections $\mathsf{wk} : \Gamma \triangleright A \to \Gamma$ and $\mathsf{var}_0 : \mathsf{Tm}\ \ (\Gamma \triangleright A)\ \ (A[\mathsf{wk}])$

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- and more...

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General theorems ensure the existence of an initial CwF
This initial model is the "syntax"

(intrinsically well-typed terms quotiented by conversion.)

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  Then, the initial CwF satisfies normalisation if any (possibly open) term of A is convertible to a normal form of of type A.

  Assuming the equality of normal forms is decidable and the proof is constructive, this is enough to obtain decidability of typechecking.

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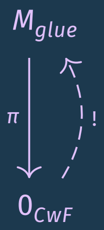

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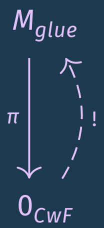

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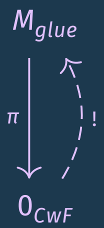

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Initiality ensures that π has a section, which associates a proof of reducibility to any object of 0_{CwF}.

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But instead of using PERs on raw syntax, we use proof-relevant predicates on well-typed syntax quotiented by conversion.

The resulting proof is arguably more principled and cleaner…at least on paper!

III.

Gluing in a proof assistant

Extensionality in type theory

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match (funext *e*) with
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match (funext e) with
| refl => t $\rightsquigarrow$???

However, we really wanted to extract an algorithm from our proof!

Extensionality in type theory

No reason to panic: in 2025, this is not an insurmountable problem anymore. We have several options:

- Cubical type theory (Coquand, Cohen, Huber, Mörtberg '16)
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Furthermore, OTT is available in Coq/Rocq (P., Leray, Tabareau) and can be implemented in Agda using rewriting rules.

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```
Con    : Set i
Sub    : Con → Con → Set j
_∘_    : ∀{Γ Δ} → Sub Δ Γ → ∀{Θ} → Sub Θ Δ → Sub Θ Γ
ass    : ∀{Γ Δ}{γ : Sub Δ Γ}{Θ}{δ : Sub Θ Δ}{Ξ}{θ : Sub Ξ Θ} → ((γ ∘ δ) ∘ θ) ~ (γ ∘ (δ ∘ θ))
id     : ∀{Γ} → Sub Γ Γ
idl    : ∀{Γ Δ}{γ : Sub Δ Γ} → (id ∘ γ) ~ γ
idr    : ∀{Γ Δ}{γ : Sub Δ Γ} → (γ ∘ id) ~ γ
◇      : Con
ε      : ∀{Γ} → Sub Γ ◇
◇η     : ∀{Γ}{σ : Sub Γ ◇} → σ ~ (ε {Γ})

Ty     : Con → Set k
_[_]T : ∀{Γ} → Ty Γ → ∀{Δ} → Sub Δ Γ → Ty Δ
[∘]T   : ∀{Γ}{A : Ty Γ}{Δ}{γ : Sub Δ Γ}{Θ}{δ : Sub Θ Δ} → A [ γ ∘ δ ]T ~ A [ γ ]T [ δ ]T
[id]T : ∀{Γ}{A : Ty Γ} → A [ id ]T ~ A

Tm     : (Γ : Con) → Ty Γ → Set l
_[_]t : ∀{Γ}{A : Ty Γ} → Tm Γ A → ∀{Δ}(γ : Sub Δ Γ) → Tm Δ (A [ γ ]T)
[∘]t   : ∀{Γ}{A : Ty Γ}{a : Tm Γ A}{Δ}{γ : Sub Δ Γ}{Θ}{δ : Sub Θ Δ} → a [ γ ∘ δ ]t ~ a [ γ ]t [ δ ]t
[id]t : ∀{Γ}{A : Ty Γ}{a : Tm Γ A} → a [ id ]t ~ a

_▹_    : (Γ : Con) → Ty Γ → Con
_,[_]_: ∀{Γ Δ}(γ : Sub Δ Γ) → ∀ {A A'} → A [ γ ]T ~ A' → Tm Δ A' → Sub Δ (Γ ▹ A)
p      : ∀{Γ A} → Sub (Γ ▹ A) Γ
q      : ∀{Γ A} → Tm (Γ ▹ A) (A [ p ]T)
▹β₁    : ∀{Γ Δ}{γ : Sub Δ Γ}{A}{a : Tm Δ (A [ γ ]T)} → p ∘ (γ ,[ ~refl ] a) ~ γ
▹β₂    : ∀{Γ Δ}{γ : Sub Δ Γ}{A}{a : Tm Δ (A [ γ ]T)} → q [ γ ,[ ~refl ] a ]t ~ a
▹η     : ∀{Γ Δ A}{γa : Sub Δ (Γ ▹ A)} → ((p ∘ γa) ,[ [∘]T ] (q [ γa ]t)) ~ γa
```

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Then, we want to define indexed CwFs over the initial CwF
→ another, even more complex, list of fields

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→ another, even more complex, list of fields

Finally, we want to define the glued model as an indexed CwF
→ welcome to transport hell!

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Most substitution laws become definitional equalities

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Most substitution laws become definitional equalities

$$(\Pi\ A\ B)[\sigma] \equiv \Pi\ \ (A[\sigma])\ \ (B[\sigma \uparrow])$$

But in our QIIT formulation, substitutions are part of the algebra signature, and we only get propositional equalities

$$(\Pi\ A\ B)[\sigma] = \Pi\ \ (A[\sigma])\ \ (B[\sigma \uparrow])$$

Gluing in a proof assistant

In conclusion, normalisation by gluing is even less tractable than old fashioned normalisation proofs.

IV.

Strictification

From propositional to definitional

Point of today's talk:
give an alternative definition of the initial CwF, for which almost all of the administrative equations become definitional equalities.

Strictifying groups

Suppose G is a group:

$$
\begin{array}{lll}
G & : & Set \\
\_\times\_ & : & G \to G \to G \\
inv & : & G \to G \\
e & : & G
\end{array}
\qquad
\begin{array}{lll}
unit_l & : & \forall x,\ e \times x = x \\
unit_r & : & \forall x,\ x \times e = x \\
inv_l & : & \forall x,\ (inv\ x) \times x = e \\
inv_r & : & \forall x,\ x \times (inv\ x) = e
\end{array}
$$

$$assoc : \forall x\ y\ z,\ (x \times y) \times z = x \times (y \times z)$$

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$$assoc : \forall x\ y\ z,\ (x \times y) \times z = x \times (y \times z)$$

Then G embeds in the group of permutations of G (Cayley's theorem)

$$Perm(G) := \{\, f : G \to G \mid \text{isBijective } f \,\}$$

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$$Perm(G) := \{\ f : G \to G \mid \text{isBijective}\ f\ \}$$

Essential point: the group law on Perm(G) is given by function composition, which is definitionally associative and unital!

Strictifying groups

If we have access to a sort of proof-irrelevant propositions, we can define a group that is isomorphic to G:

$$G' := \{\, f : G \to G \mid \exists (g : G), f = \tau_g \,\}$$

With G' being definitionally associative and unital:

$$((f, f_\varepsilon) \circ (g, g_\varepsilon)) \circ (h, h_\varepsilon) \equiv (f, f_\varepsilon) \circ ((g, g_\varepsilon) \circ (h, h_\varepsilon))$$
$$(f, f_\varepsilon) \circ (id, id_\varepsilon) \equiv (f, f_\varepsilon)$$
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Strictifying CwFs, first attempt

Cayley's theorem is an instance of the Yoneda lemma:
any category $\mathcal{C}$ embeds into the category $\hat{\mathcal{C}} := \mathrm{Hom}(\mathcal{C}^{op}, \mathrm{Set})$.

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The Yoneda generalises to categories with families:

Given a CwF $\mathcal{C}$, the presheaf category $\hat{\mathcal{C}}$ is naturally equipped with a CwF structure inherited from Set. Additionally, there is an embedding of CwFs $\mathcal{C} \to \hat{\mathcal{C}}$.

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We can thus try the same trick: define $\mathcal{C}'$ to be the image of $\mathcal{C}$ under the embedding.

Strictifying CwFs, first attempt

$\mathcal{C}'$ is thus isomorphic to $\mathcal{C}$, and enjoys more definitional eqs:

- substitutions are definitionally associative
- substitutions are definitionally unital
- wk and var_0 satisfy their equations definitionally

Strictifying CwFs, first attempt

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...BUT

The commutation of substitutions with binders is not definitional

$$(\Pi\ A\ B)[\sigma] \not\equiv \Pi\ (A[\sigma])\ (B[\sigma \uparrow])$$

Strict presheaves

Unfolding the computations, the reason why substitutions do not commute with binders boils down to natural transformations not being definitional

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In "Russian constructivism in a prefascist theory" (2020), Pédrot introduces prefascist sets, an alternative definition of presheaves that is strictly natural.

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If we reproduce our strictification construction using Pédrot's definition, we obtain a new CwF $\mathcal{C}''$, in which all* the administrative equalities are definitional.

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If we reproduce our strictification construction using Pédrot's definition, we obtain a new CwF $\mathcal{C}''$, in which all* the administrative equalities are definitional.

We formalised the construction of $\mathcal{C}''$ and its isomorphism with $\mathcal{C}$ in Agda. Surprisingly doable, even when the CwF is equipped with dependent products and booleans!

Back to our original goal

Applying our strictification construction to the initial CwF, it becomes much easier to construct gluing models. We were able to define a canonicity model (which computes normal forms for closed terms) in about 200 lines!

(the strictification construction is about 4000 lines)

Thank you!