Rules of Simple Type Theory

1 Types

A type is either...

- $A \to B$ (assuming A and B are types),
- $A \times B$ (assuming A and B are types),
- $A + B$ (assuming A and B are types),
- the empty type $\bot$,
- the unit type $\top$,
- or a base type (we will only consider $\mathbb{N}$ as a base type).

2 Typing rules

Function types

$$\textsc{Arr-Intro}\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda\, x\,.\, t : A \to B} \qquad \textsc{Arr-Elim}\ \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

Computation rules:

- $(\lambda\, x\,.\, t)\ u \equiv t[u/x]$
- $t \equiv \lambda\, x\,.\,(t\ x) \qquad$ (when t has type $A \to B$)

Product types

$$\textsc{Prod-Intro}\ \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \langle\, t\,, u\,\rangle : A \times B} \qquad \textsc{Prod-Elim1}\ \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_1(t) : A} \qquad \textsc{Prod-Elim2}\ \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_2(t) : B}$$

Computation rules:

- $\pi_1(\langle\, t\,, u\,\rangle) \equiv t$
- $\pi_2(\langle\, t\,, u\,\rangle) \equiv u$
- $t \equiv \langle\, \pi_1(t)\,, \pi_2(t)\,\rangle \qquad$ (when t has type $A \times B$)

Sum types

$$\textsc{Sum-Intro1}\ \frac{\Gamma \vdash t : A}{\Gamma \vdash \mathrm{in}_1(t) : A + B} \qquad \textsc{Sum-Intro2}\ \frac{\Gamma \vdash t : B}{\Gamma \vdash \mathrm{in}_2(t) : A + B}$$

$$\textsc{Sum-Elim}\ \frac{\Gamma, x : A \vdash t : C \qquad \Gamma, x : B \vdash u : C \qquad \Gamma \vdash v : A + B}{\Gamma \vdash \begin{array}{l} \text{match } v \text{ with} \\ \mid \mathrm{in}_1(x) \mapsto t \\ \mid \mathrm{in}_2(x) \mapsto u \end{array} : C}$$

Computation rules:

- match $\mathrm{in}_1(v)$ with $\mid \mathrm{in}_1(x) \mapsto t \mid \mathrm{in}_2(x) \mapsto u \quad \equiv \quad t[v/x]$
- match $\mathrm{in}_2(v)$ with $\mid \mathrm{in}_1(x) \mapsto t \mid \mathrm{in}_2(x) \mapsto u \quad \equiv \quad u[v/x]$

Empty type

$$\frac{\Gamma \vdash t : \bot}{\Gamma \vdash \mathrm{raise}(t) : A}\ \textsc{Empty-Elim}$$

No computation rules!

Unit type

$$\frac{}{\Gamma \vdash \star : \top}\ \textsc{Unit-Intro}$$

Computation rule: $\quad t \equiv \star \qquad$ (when t has type $\top$)

Natural numbers

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}}\ \textsc{Nat-Intro1} \qquad \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathrm{suc}(t) : \mathbb{N}}\ \textsc{Nat-Intro2} \qquad \frac{\Gamma \vdash t : \mathbb{N} \qquad \Gamma \vdash u : \mathbb{N}}{\Gamma \vdash t + u : \mathbb{N}}\ \textsc{Nat-Sum}$$

Computation rules:

- $0 + t \equiv t$
- $\mathrm{suc}(t) + u \equiv \mathrm{suc}(t + u)$