Exercises – Introduction to Type Theory

Exercise 4 – Dependent Types

1. The classical statement of Fermat's last theorem is

   There are no three positive integers a, b, c such that $a^n + b^n = c^n$ when $n \geq 3$.

   Write down a type which encodes the statement of Fermat's last theorem.

2. Assume that A and B are types and $R : A \to B \to \mathsf{Type}$ is a relation. Find a term t which admits the following type:

   $$\big(\Pi\,(x : A)\,.\,\Sigma\,(y : B)\,.\,R\ x\ y\big) \to \big(\Sigma\,(f : A \to B)\,.\,\Pi\,(x : A)\,.\,R\ x\ (f\ x)\big)$$

   To draw a parallel with classical mathematics, replace Π with $\forall$ and Σ with $\exists$. Do you recognise the statement that you just proved?

Exercise 5 – Natural numbers

1. Use the recursor of $\mathbb{N}$ to define addition as a function $\mathsf{plus} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$.

2. Use the recursor of $\mathbb{N}$ to define a binary relation $\mathsf{gt} : \mathbb{N} \to \mathbb{N} \to \mathsf{Type}$ which encodes the (strict) order relation on $\mathbb{N}$. Can you think of another definition?

3. Find a proof term for the statement $\Pi\,(n : \mathbb{N})\,.\,\mathsf{gt}\ (\mathsf{plus}\ 1\ n)\ n$.

Exercise 6 – Martin-Löf's equality type

Assume that A is a type, and that x, y, z are three terms of type A.

1. Find a proof term for the statement $(x = y) \to (y = x)$.

2. Find a proof term for the statement $(x = y) \to (y = z) \to (x = z)$.

3. ($\star$) Find a proof term for $\Pi\,(n : \mathbb{N})\,.\,\Pi\,(m : \mathbb{N})\,.\,\mathsf{plus}\ n\ m = \mathsf{plus}\ m\ n$.
   Hint: you might want to define an auxiliary lemma.