A Constructive Cellular Approximation Theorem in HoTT

Axel Ljungström and Loïc Pujet

Stockholm University, Stockholm, Sweden

At the heart of homotopy type theory (HoTT) is the analogy between types and spaces. This permits the use of type theory as a language for algebraic topology, *i.e.* for the study of spaces and maps between spaces up to homotopy by means of algebraic invariants, such as homotopy groups [13, 1, 4] and (co)homology groups [9, 5, 3, 14, 7, 2, 6, 8, 10]. Although the methods of algebraic topology apply to very general notions of spaces, the theory is often easier to develop in the context of a more restricted and well-behaved class: CW complexes. As such, it is natural to define CW complexes in the language of HoTT, in order to obtain a notion of spaces which is easier to work with than arbitrary types.

In this work, we revisit the definition of CW complexes given by Buchholtz and Favonia [3] and develop their theory. In particular, we focus on the *cellular approximation theorem*, a cornerstone result in algebraic topology which says that arbitrary maps between CW complexes and their homotopies may be approximated by maps and homotopies which respect the cellular structure [12, chap. 10]. We give a constructive proof of the theorem which relies heavily on the (provable) principle of finite choice[1], and we discuss the extent to which the theorem can be strengthened while remaining constructive. The work we present here is intended to serve as a foundation for a larger project on the development of cellular homology with Anders Mörtberg.

In order to define CW complexes, we will need the following definition:

Definition 1 (CW skeleta)**.** *An **ordered CW skeleton** is an infinite sequence of types*

$$\varnothing = C_{-1} \xrightarrow{\iota_{-1}} C_0 \xrightarrow{\iota_0} C_1 \xrightarrow{\iota_1} \dots$$

equipped with maps $\alpha : \mathbb{S}^n \times A_n \to C_n$ where A_n is equivalent to $\mathsf{Fin}(k_n)$ for some $k_n : \mathbb{N}$ and the following square is a pushout:

$$\begin{array}{ccc} \mathbb{S}^n \times A_n & \longrightarrow & A_n \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow \\ C_n & \xrightarrow[\iota_n]{} & C_{n+1} \end{array}$$

*An **unordered CW skeleton** is defined similarly, but each A_n is only assumed to be merely finite, i.e. for all n we have a proof of $\|A_n \simeq \mathsf{Fin}(k_n)\|_{-1}$.*

The pushout condition ensures that the $(n+1)$-skeleton C_{n+1} is obtained by attaching a finite number of n-dimensional cells to the n-skeleton C_n. In the case of an ordered CW skeleton, each type of cells is equipped with an order inherited from $\mathsf{Fin}(k_n)$, hence the name. Given a CW skeleton $C_\bullet$, we write C_∞ for the colimit of the sequence of n-skeleta, and for any n we write $\iota_\infty : C_n \to C_\infty$ for the inclusion of C_n into the colimit C_∞.

Definition 2 (CW complexes)**.** *A type X is said to be an ordered (resp. unordered) CW complex if there merely exists an ordered (resp. unordered) CW skeleton $C_\bullet$ such that X is equivalent to the colimit C_∞.*

[1] The proof has been fully formalised in `Cubical Agda`, and is available at https://github.com/loic-p/cellular/blob/main/summary.agda

A map between two CW complexes X and Y is simply a map between the underlying types. The cellular approximation theorem states that such maps may be approximated by *cellular maps*, *i.e.* sequences of maps between the n-skeleta of X and Y. In order to prove this theorem constructively for unordered CW complexes, we need to define finite cellular maps:

Definition 3 (Cellular m-maps)**.** *Given two CW skeleta $C_\bullet$ and $D_\bullet$, a cellular m-map from $C_\bullet$ to $D_\bullet$ is a finite sequence of maps $(f_n : C_n \to D_n)_{n\leq m}$ equipped with a family of homotopies $h_n(x) : (\iota_n \circ f_n)(x) = (f_{n+1} \circ \iota_n)(x)$ for $n < m$.*

Definition 4 (Cellular m-homotopies)**.** *Given two cellular m-maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, an m-homotopy between $f_\bullet$ and $g_\bullet$, denoted $f_\bullet \sim_m g_\bullet$, is a finite sequence of homotopies $(h_n : \iota_n \circ f_n = \iota_n \circ g_n)_{n\leq m}$ such that for $n < m$ and $x : C_n$ the following square commutes:*

$$\begin{array}{ccc} (\iota_{n+1} \circ f_{n+1} \circ \iota_n)\ x & \overset{(h_{n+1}\circ\iota_n)(x)}{=\!=\!=\!=} & (\iota_{n+1} \circ g_{n+1} \circ \iota_n)\ x \\ \| & & \| \\ (\iota_{n+1} \circ \iota_n \circ f_n)\ x & \underset{(\iota_{n+1}\circ h_n)(x)}{=\!=\!=\!=} & (\iota_{n+1} \circ \iota_n \circ g_n)\ x \end{array}$$

Theorem 1 (Cellular m-approximation theorem)**.** *Given two unordered CW skeleta $C_\bullet, D_\bullet$, a map $f : C_\infty \to D_\infty$ and $m : \mathbb{N}$, there merely exists an m-cellular map $f_\bullet : C_\bullet \to D_\bullet$ such that $\iota_\infty \circ f_m = f \circ \iota_\infty$.*

Theorem 2 (Cellular m-approximation theorem, part 2)**.** *Let $C_\bullet, D_\bullet$ be unordered CW skeleta and consider two cellular m-maps map $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ with a such that $f_m = g_m$. In this case, there merely exists a cellular m-homotopy $f_\bullet \sim_m g_\bullet$.*

Sketch of proofs. The proof of Theorem 1 is done by induction on m: if we have an n-approximation of f, we can use the principle of finite choice to obtain the mere existence of an $(n + 1)$-approximation. Note that we only approximate f up to a fixed dimension m, so that the construction only needs finitely many calls to finite choice, which is constructively valid [13, exercise 3.22]. Theorem 2 is proved using the exact same techniques. ☐

Although our statements of the cellular approximation theorems are sufficient to develop cellular homology in HoTT [11], they are weaker than their classical counterparts on two points. Firstly, we only obtain the *mere existence* of an approximation. However, since every construction in HoTT has to be homotopy invariant, the untruncated version of Theorem 1 is actually inconsistent: when specialised to the unit type and the circle (which are both CW complexes), the untruncated approximation theorem amounts to the contractibility of the circle. Therefore, some amount of truncation is required to state the theorem in HoTT. Secondly, the classical cellular approximation theorems are stated for $m = \infty$, while ours only provide finite approximations. In fact, due to the fundamental reliance of the theorem on finite choice, we conjecture that the case $m = \infty$ is equivalent to the axiom of countable choice, and thus not provable in constructive HoTT.

Conjecture 1. *The case $m = \infty$ of the cellular approximation theorems is not provable in plain HoTT for unordered CW skeleta.*

However, in the case of *ordered* CW skeleta, we expect that it is possible to use the order on the sets of cells to pick a *minimal* approximation at each stage for some carefully defined order. This eschews the need for finite choice and thus we conjecture the following:

Conjecture 2. *The cellular approximation theorems hold for $m = \infty$ for ordered CW skeleta.*

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