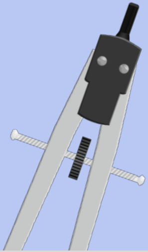
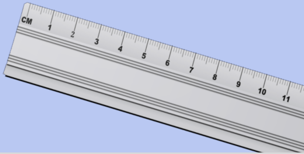
A CONSTRUCTIVE cellular approximation theorem





Axel Ljungström, Loïc Pujet

12 june 2024

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But how can we prove that two types are not equal?

Analysis Situs Abridged

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→ equivalent spaces are given isomorphic algebraic invariants

→ the difficult problem of telling spaces apart is reduced to the (hopefully) easier problem of telling algebraic gadgets apart

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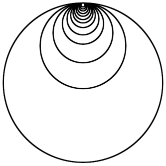

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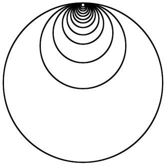

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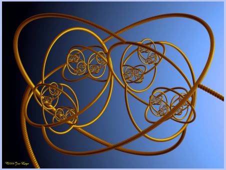

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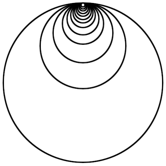

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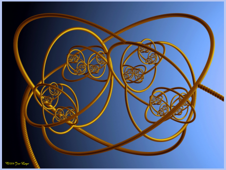

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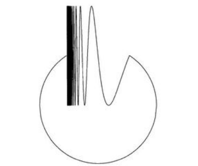

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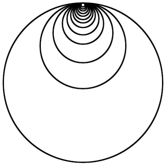

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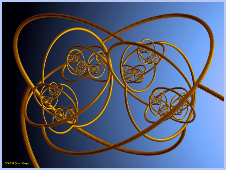

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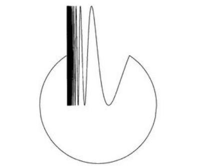

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Good news: there is a very comprehensive category of topological spaces where algebraic invariants are very effective: CW complexes

CW complexes, type-theoretically

Following Buchholtz and Favonia '18, we construct them iteratively:

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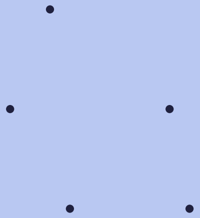

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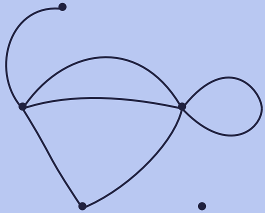

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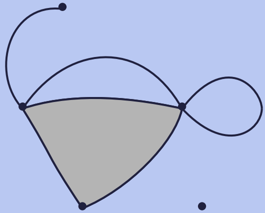

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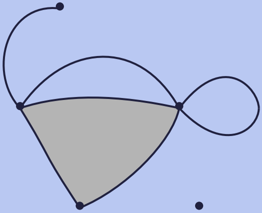
$$\begin{array}{ccc} A_2 \times \mathbb{S}^1 & \xrightarrow{fst} & A_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$



Spooky scary CW skeletons

Thus, we define a CW skeleton to be given by the following data:

- a family of types X indexed by $\mathbb{N} \cup \{-1\}$
- a number of cells for every dimension, given by $f : \mathbb{N} \to \mathbb{N}$
- an attaching map for every dimension

$$\alpha_n : Fin\,(f\ n) \to \mathbb{S}^n \to X_{n-1}$$

Such that X_{-1} is empty and that for all n, this diagram is a pushout:

$$\begin{array}{ccc} Fin\,(f\ n) \times \mathbb{S}^n & \longrightarrow & Fin\,(f\ n) \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

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Then, we define a morphism of CW skeleta as a family of maps

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such that every square commutes up to homotopy.

This induces a map between the colimits

$$f_\infty : X_\infty \to Y_\infty$$

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Finally, we define a type to be a CW complex if it is merely equivalent to the colimit of a CW skeleton.

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Now, how should we define maps between two CW complexes? Maybe a map that is the colimit of some map of CW skeleta? We can do simpler!

The Cellular Approximation Theorem

Cellular approximation theorem : let X and Y be two CW skeleta. Every map from X_∞ to Y_∞ is the colimit of a map of CW skeleta.

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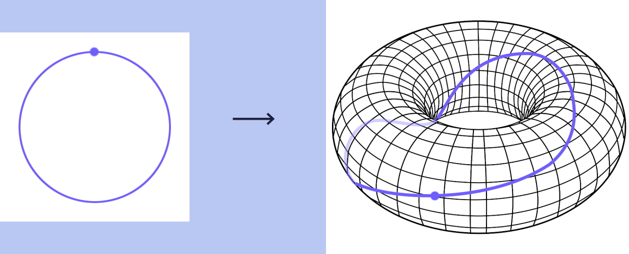

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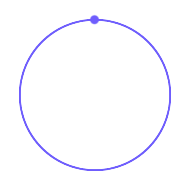

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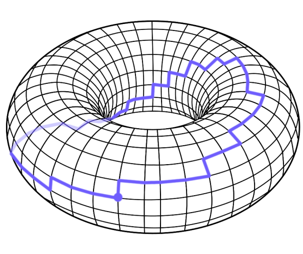

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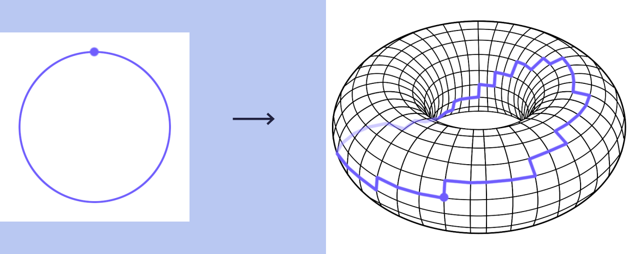

Likewise, any homotopy can be approximated by a homotopy that respects the cellular structure

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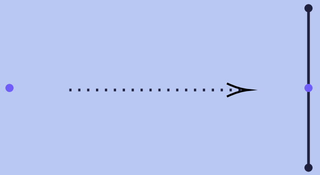

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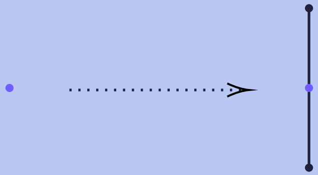

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And since there is an infinite number of dimensions, there is an infinite number of dependent choices to make. ☹

Laziness to the rescue

Our main goal with this theorem is to define a functorial theory of cellular homology.

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It turns out that for any given n, the definition of $H_n(X)$ only depends on the $(n+1)$-skeleton of X. And we only need finite choice to prove the cellular approximation theorem up to dimension $n+1$!

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- Lift the definition to the category of CW complexes
- (WIP) Prove the Eilenberg–Steenrod axioms, the Hurewicz theorem...

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Thus, in dimension 0, we can easily approximate a point by the smallest element of the 0-skeleton in its connected component.

In dimension 1, it gets more difficult: even with a finite number of 1-cells, we can have an infinite number of cellular maps (remember that $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$)

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But

Theorem: if we modify the definition of CW skeleta so that the sets of cells are unordered, then the full cellular approximation theorem implies a weak form of choice.

Thank you!